Classification of C*-Algebras

Bruce Blackadar
Department of Mathematics and Statistics
University of Nevada, Reno
bruceb@unr.edu
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PREFACE

This book is an attempt to give a comprehensive account of what I consider to be the greatest recent achievement in the subject of operator algebras, and indeed one of the most significant recent advances in all of mathematics: the complete classification of a large class of simple C*-algebras by a manageable and computable invariant. I have tried to present a unified treatment of a result and story which has been developed and told only piecemeal over a long period of time; my goal is not only to give a complete treatment, but also one which is largely accessible and readable by nonspecialists and explains the nature and significance of the result to a wide audience.

There have been at least three previous books on the classification program, [RS02], [Lin01b], and [Lin17] ([BO08] is also relevant but not exactly on the same program), as well as some survey articles ([**?**], [ET08]), but these were mostly written before the program was completed and do not tell the full story. Nonetheless, they are valuable references which in particular were very useful in the preparation of this book. There is one survey article, [**?**], which was written after the conclusion of the project (unital version) by one of the big names of the second half and which contains especially noteworthy philosophical insights.

I have had to make a delicate balance between making the book reasonably self-contained and not reinventing the wheel. I have briefly plagiarized some of my own previous writings (occasionally with amplification or modernization) on topics which are essential tools in the classification program, but I have generally just made reference to other works, especially ones easily available online. References tend to frequently be to my own previous books, where things are usually written out in their proper form (at least according to the way I think). Most of these can also be found in other standard references, which may be more to the reader's taste.

I have tried to write a complete and relatively objective account of the classification program, but I cannot be completely objective since I have been personally involved at several stages (although I certainly could not claim to be one of the major players in the program). I have of necessity presented things according to how I have viewed and understood the program myself, although I have of course been heavily informed and influenced by philosophies, points of view, and words of wisdom imparted to me by many people over the years. This is what any mathematician writing a book on any subject should, and probably must, do, since that is the unique contribution he or she can make; someone else writing on the same subject will have a somewhat different perspective. If I have put too much emphasis on parts of the classification I have personally worked on, I hope it will be understood that these are simply the parts I know best and perhaps can describe most intelligently. If another participant attempts a similar volume (as I hope some will), each personal perspective will differ and add to the whole picture.

Contents

Chapter I

Introduction and Overview

The complete classification of a large class of simple C*-algebras was recently concluded (the unital case in 2015, nonunital case in 2019-22). This is an achievement comparable to the classification of finite simple groups announced in the 1980s, which was widely (and rightly) regarded as one of the great accomplishments of twentieth century mathematics (although not actually completed until 2004).

The C*-algebra classification theorem has so far not received such widespread acclaim, possibly because C*-algebras are not as familiar as finite groups, and possibly for other reasons too. The object of this book is to give an exposition of the classification theorem and why it is interesting and important, and to describe the complicated path to its proof. Depth and sophistication of the mathematics involved will gradually increase throughout the book, beginning at a level which we hope will be accessible to a general mathematical audience; the goal will not simply be to give a unified proof of the theorem, but also to put the pieces into a natural context.

The (abbreviated) statement of the classification theorem is:

Theorem. The Elliott invariant is a complete isomorphism invariant for regular separable non-elementary simple UCT C*-algebras: if A and B are regular separable non-elementary simple UCT C*-algebras, and $Ell(A) \cong Ell(B)$, then $A \cong B$. In fact, $A \mapsto Ell(A)$ is a functor, and any isomorphism of $Ell(A)$ and $Ell(B)$ is induced by an isomorphism of A and B.

The terms used in the statement will be gradually introduced and explained; see VIII.6.2.1. for a full statement of the Theorem.

As a companion result complementing the Classification Theorem, we will also give a clean and (deceptively) simple characterization of all the values the Elliott invariant (which is essentially K-theory) can take on the classifiable algebras.

There is (at least) one significant difference between the group and C*-algebra results. One of the biggest difficulties in the finite group case is the existence of sporadic simple groups, and a large part of the complexity of the group classification theorem is in describing the list of sporadic groups and showing the list is complete. But there are no sporadic classifiable simple C*-algebras (unless one counts the elementary C*-algebras, which are easily described and classified, cf. I.1.5.13., and which are perhaps actually the rough analog of the finite cyclic groups of prime order): a single clean invariant works in every case, and as a result the statement of the classification theorem is vastly simpler.

The C*-algebra classification theorem has another feature in common with the finite group classification: both were projects which were the work of many hands which gradually took final form over many years. It was estimated that the finite group classification, as it existed when announced, comprised hundreds of research papers totaling about 15,000 printed pages (newer consolidated expositions have reduced it to roughly 3000 pages). Similarly, the C*-algebra classification theorem was developed in many stages over about forty years, with some preliminary steps going back much farther, and if the theorem were to have been published in a single blockbuster paper, I estimate it would have between 25 and 50 coauthors (depending how work contributing or leading to the theorem is counted). I have not tried to estimate the page count of the proof of the unital case as it exists in the published literature, but I am confident it is more than 1000. Since the proof of the unital case was completed, parts have been streamlined and simplified, and the new approach has the additional advantage that it works with relatively little modification in the nonunital case also; we have incorporated the simplifications (so far) in our exposition.

The development of the Classification Theorem has roughly divided into several parts (dates are approximate):

Construction of examples and counterexamples (1959-2017).

Classification of simple unital ASH algebras with slow dimension growth ((1959-)1973-1996).

Classification of purely infinite UCT C*-algebras (1993-2000).

Classification of regular stably finite simple unital UCT C*-algebras (1998-2015).

Classification of regular stably projectionless simple UCT C*-algebras (2012-2022).

Although many people have made major contributions to the theorem, one name stands out: George A. Elliott. He has not only been centrally involved in the project from beginning to end, but it was really his brainchild: he had the vision and courage to believe a classification theorem via K-theory was possible long before almost anyone else (the classification program is often called the "Elliott program"). And he was able to see to a successful conclusion a program many people thought was folly. George can now have the last laugh.

I.1. C*-algebras

I.1.1. General Description

I.1.1.1. C*-Algebras are mathematical objects with a rich and beautiful structure, both algebraic and topological. The formal definition is given in II.1.4.1.. The components of the structure are:

A vector space structure over the complex numbers $\mathbb{C}$.

A multiplication giving a ring (algebra) structure, not necessarily commutative or unital.

An involution (adjoint) operation, denoted $x \mapsto x^*$ and sometimes called the *star operation*.

A complete norm giving a (complete) metric and thus a topology.

I.1.1.2. There is also a delicate interaction between the algebraic and topological structures, each dictating properties of the other and together giving considerable rigidity. Some of the important properties are:

(i) A homomorphism (*-homomorphism) between C*-algebras is norm-decreasing, and its image is closed (II.1.5.6., [Bla06, II.5.1.2]). [We will take "homomorphism" to always mean "$\mathbb{C}$-linear *-homomorphism": these are the natural morphisms for the category of C*-algebras. Similarly, "isomorphism" will mean "$\mathbb{C}$-linear *-isomorphism."]

(ii) An injective homomorphism (*-homomorphism) between C*-algebras is an isometry (II.1.5.7.). In particular, the norm on a C*-algebra is unique; in fact, it is completely determined by the algebraic structure (II.1.5.5.).

(iii) Every (topologically) closed two-sided ideal in a C*-algebra is closed under scalar multiplication and under the adjoint (a **-ideal*), and the quotient is a C*-algebra under the quotient norm [Bla06, II.5.1.1]. [We will take "ideal" to mean "two-sided ideal" unless otherwise specified; "closed ideal" will mean topologically closed.]

(iv) Although a C*-algebra need not be unital (have a multiplicative identity), it always has a well-behaved approximate unit [Bla06, II.4.1.4].

(v) If A is a C*-algebra, the set $A_+ = \{x^*x : x \in A\}$ is a closed cone in A, called the *positive cone*, and A_+ defines a partial order on A with nice properties (II.1.7.3.). Every positive element a has a unique positive square root, denoted $a^{1/2}$ (II.1.7.2.(vii); this is a special case of *functional calculus* (II.1.6.1.)).

These are in addition to properties valid in general Banach algebras, such as

(vi) Addition and multiplication are jointly continuous (the involution is isometric, thus continuous too).

(vii) If A is a unital C*-algebra, and $x \in A$, $\|1 - x\| < 1$, then x is invertible. In particular, the set (group) of invertible elements in A is an open set.

(viii) If A is a unital C*-algebra and J is a proper ideal of A, then the closure of J is also a proper ideal of A. (The closure of any *-subalgebra of A is a C*-subalgebra of A.)

(ix) If A is a unital C*-algebra and $x \in A$, then the *spectrum*

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible in } A\}$$

is a nonempty compact subset of $\mathbb{C}$.

There are many additional properties of the spectrum of elements of a C*-algebra which are discussed in ().

I.1.1.3. For (many) technical reasons, we will work exclusively over the complex numbers. There is a similar, but more complicated and less clean, theory of real C*-algebras which we will not discuss. A classification for real C*-algebras along the lines of the classification described in this book has begun, cf. [**?**], [**?**], [**?**], ().

I.1.1.4. Throughout much of this introduction, we will almost exclusively focus attention on unital C*-algebras. Much of what we do will apply to nonunital C*-algebras also, but often with considerable additional technicalities (we will briefly discuss the modifications needed in the nonunital case in the introduction, and the nonunital case will be done in full later in the book). When we can handle nonunital C*-algebras on an equal basis, we will, but we will not hesitate (in the introduction) to retreat to the unital case when it is simpler or cleaner. When dealing with unital C*-algebras, we will almost exclusively consider only unital homomorphisms (*-homomorphisms). We will also assume the Axiom of Choice throughout (i.e. we work in ZFC); the AC will be largely unnecessary when we stick to separable C*-algebras.

I.1.2. Standard Examples

There are two standard classes of examples of C*-algebras:

I.1.2.1. Example. Let $\mathcal{H}$ be a Hilbert space (complete complex inner product space). Then the set $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ has a natural structure as a *-algebra (multiplication of operators is composition), and the operator norm makes $\mathcal{B}(\mathcal{H})$ into a C*-algebra. More generally, any norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ is a C*-algebra; such C*-algebras are called *concrete C*-algebras (of operators)* on $\mathcal{H}$. In particular, $\mathbb{M}_n = M_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$ is a C*-algebra with its usual algebraic operations (a^* is the usual adjoint matrix, the complex conjugate of the transpose of a) and the operator norm (which can be hard to compute but which is equivalent to any other norm on $\mathbb{M}_n$ since it is finite-dimensional).

The set $\mathcal{K}(\mathcal{H})$ of *compact operators* on $\mathcal{H}$ is a concrete C*-algebra on $\mathcal{H}$, in fact an ideal in $\mathcal{B}(\mathcal{H})$ ($\mathcal{K}(\mathcal{H})$ can be defined to be the closure of the set of finite-rank operators on $\mathcal{H}$). $\mathcal{K}(\mathcal{H})$ is nonunital if $\mathcal{H}$ is infinite-dimensional (it is all of $\mathcal{B}(\mathcal{H}) \cong \mathbb{M}_n$ if $\mathcal{H} \cong \mathbb{C}^n$). Thus a C*-algebra is not necessarily unital. Denote by $\mathbb{K}$ the C*-algebra of compact operators on a separable infinite-dimensional Hilbert space.

I.1.2.2. Example. Let X be a compact Hausdorff space. Let $C(X)$ denote the set of complex-valued continuous functions on X, with the usual pointwise addition and multiplication of functions and supremum norm. For $f \in C(X)$, let $f^* = \bar{f}$ be the complex conjugate of f. Then with these operations and norm, $C(X)$ becomes a unital commutative C*-algebra.

More generally, if X is a locally compact Hausdorff space, let $C_0(X)$ be the set of complex-valued continuous functions on X vanishing at infinity. In the same manner, $C_0(X)$ is a commutative C*-algebra, which is unital if and only if X is compact (in which case $C_0(X) = C(X)$).

Example I.1.2.2. has an extension which is relevant for later classification constructions:

I.1.2.3. Example. Let B be a C*-algebra, and X a locally compact Hausdorff space. Let $C_0(X, B)$ be the set of continuous functions $f : X \to B$ which vanish at infinity in the sense that the (continuous) function $g(x) = \|f(x)\|$ from X to $\mathbb{R}$ vanishes at infinity. Add and multiply functions pointwise, and also take the involution pointwise, i.e. $[f^*](x) = [f(x)]^*$. Give $C_0(X, B)$ the supremum norm: $\|f\| = \sup_x \|f(x)\|$. Then $C_0(X, B)$ is a C*-algebra, which is unital if and only if B is unital and X is compact. (This is a special case of the C*-algebra tensor product ().)

In particular, $C_0(X, \mathbb{M}_n)$ is a C*-algebra for any locally compact X. The C*-algebra $C_0(X, \mathbb{M}_n)$ is naturally isomorphic as a *-algebra to $M_n(C_0(X))$. Actually, a matrix algebra over any C*-algebra is a C*-algebra (this can be shown easily using I.1.2.4.(i): if A is a concrete C*-algebra of operators on $\mathcal{H}$, then

$M_n(A)$ is naturally a concrete C*-algebra of operators on $\mathcal{H}^n$), another special case of the C*-tensor product.

Examples I.1.2.1. and I.1.2.2. are universal because of the following result. This statement is actually two distinct theorems, both of which are commonly called the *Gelfand-Naimark Theorem* since they both first appeared in [GN43]; to distinguish them, (ii) is sometimes called *Gelfand's Theorem* since it is primarily due to Gelfand ([Gel41], [GfRv46]).

I.1.2.4. Theorem. [Gelfand-Naimark Theorem(s)] (i) Every C*-algebra is isomorphic (isometrically *-isomorphic) to a concrete C*-algebra of operators.

(ii) Every commutative C*-algebra is isomorphic (isometrically *-isomorphic) to $C_0(X)$ for some locally compact Hausdorff space X. The space X is unique up to homeomorphism and can be recovered from $C_0(X)$.

I.1.2.5. Because of (i), C*-algebras are often called *operator algebras*, even though an abstract C*-algebra generally has no connection with any operators. (Actually the term *operator algebra* is more general, since it can include other types of algebras isomorphic to algebras of operators, e.g. certain non-selfadjoint algebras.)

I.1.2.6. In general, a C*-algebra can be realized in many different ways as a concrete C*-algebra of operators. Description of the possible ways is called the *representation theory* of the C*-algebra. Before about the mid-1970s, C*-algebra theory consisted mostly of representation theory, which is intractable (in a precise technical sense) outside a rather small class of "almost commutative" C*-algebras called *Type I*, *postliminal*, or *GCR* C*-algebras; thus conventional wisdom was that these were the only (separable) C*-algebras that could be "understood." This point of view gradually changed and broadened: in fact, every C*-algebra to which the classification theorem applies fails to be Type I! Representation theory remains an important part of C*-algebra theory but is no longer the dominant part, and we now have a much broader sense in which C*-algebras can be "understood."

"We shall have little, and certainly nothing good, to say about NGCR C*-algebras here."

W. Arveson, 1976[1]

(An NGCR C*-algebra is one with no nonzero Type I ideal; all simple nonelementary C*-algebras are NGCR.) I include this quote merely to illustrate how thinking has changed in this subject − in the future someone may well quote one of my own statements about nonnuclear or nonexact C*-algebras in the same vein!

[1][Arv76, p. 26]. This book has a famous misprint on the cover, which reads "An Invitation to C*-Algebra." Arveson was *not* happy about it.

I.1.2.7. The actual statement of I.1.2.4.(ii) is more precise: there is a category equivalence. In the unital case, it states that the correspondence $X \leftrightarrow C(X)$ is a contravariant category equivalence between the category of compact Hausdorff spaces and continuous functions and the category of unital commutative C*-algebras and unital *-homomorphisms (see [Bla06, II.2.2.7] for the nonunital version). Thus a modern point of view is that C*-algebras are "noncommutative topological spaces" and that C*-algebra theory is *noncommutative topology* (to paraphrase P. T. Johnstone, noncommutative topology could be called "the art of pointless thinking.") This point of view has been very powerful and valuable in expanding the scope of the theory and its applications, and in providing tools such as K-theory; see e.g. the introduction to [Con94].

I.1.2.8. However, the C*-algebras discussed in this book will mostly bear much closer similarity to the $\mathcal{B}(\mathcal{H})$ example than the $C(X)$ examples, although topology will come in at many crucial steps in the analysis. In fact, it is a good guide to regard $\mathbb{M}_n$ as a model for simple C*-algebras: in many respects a general simple unital C*-algebra will just be an infinite-dimensional version of a matrix algebra. Similarly, $\mathbb{K}$ can be regarded as a (not quite as good) model for nonunital simple C*-algebras.

I.1.3. Applications

Group Representations

I.1.3.1. A historically important application is to group representations (more precisely, unitary representations of locally compact topological groups). If G is a locally compact topological group, there is an associated C*-algebra $C^*(G)$ which has the same unitary representation theory as G, i.e. there is a natural one-one correspondence between the strongly continuous unitary representations of G and the representations of $C^*(G)$. $C^*(G)$ is made by forming the Banach *-algebra $L^1(G)$ using convolution and Haar measure on G and completing $L^1(G)$ with respect to another norm induced by the representations of G; thus the representation of $C^*(G)$ corresponding to a representation π of G is an "integrated form" of π. The group C*-algebra is an important tool in representation theory. This construction applies to any group with no topology, since it can be considered to be a locally compact group with the discrete topology. $C^*(G)$ is unital if and only if G is discrete.

If G is a locally compact abelian group, then $C^*(G)$ is commutative (and, conversely, if $C^*(G)$ is commutative, then G is abelian). Thus by I.1.2.4.(ii) $C^*(G) \cong C_0(X)$ for some X. In fact, $C^*(G) \cong C_0(\hat{G})$, where $\hat{G}$ is the Pontrjagin dual of G. C*-algebra theory is an efficient way to develop the duality theory of locally compact abelian groups which underlies harmonic analysis. The theory of unitary representations of locally compact groups, not necessarily abelian, is often called *abstract harmonic analysis* (cf. [HR79]). A good historical survey of the connections between C*-algebras and group representations is [Ros94b].

Crossed Products and Dynamical Systems

I.1.3.2. More generally, if A is a C*-algebra, G is a locally compact topological group, and α is a homomorphism from G to the group $Aut(A)$ of automorphisms of A (*-isomorphisms from A onto A) which is point-norm continuous, i.e. $g \mapsto \alpha_g(x)$ is a continuous function from G to A for all $x \in A$ (this condition is automatic if G is discrete), then there is a construction of a *crossed product* $A \rtimes_\alpha G$ encoding A, G, and α, roughly expanding A to a larger C*-algebra in which the automorphisms become inner (this is strictly correct only if A is unital and G is discrete, so the crossed product is unital; in general A and G only sit in the "multiplier algebra" () of $A \rtimes_\alpha G$). The group C*-algebra $C^*(G)$ is the special case where $A = \mathbb{C}$ and the action α is (necessarily) trivial.

I.1.3.3. An important special case of the crossed product is the case where A is commutative, $A = C_0(X)$. A point-norm continuous action of a locally compact group G on $C_0(X)$ is the same thing as a continuous action of G on X by homeomorphisms, i.e. $(g, x) \mapsto \alpha_g(x)$ is a continuous function from $G \times X$ to X (the condition is again automatic if G is discrete). Such an action is called a *dynamical system*, especially if G is $\mathbb{Z}$ or $\mathbb{R}$. The crossed product $C_0(X) \rtimes_\alpha G$ is called the *transformation group C*-algebra* of the dynamical system. Transformation group C*-algebras have long been an important tool in studying dynamical systems; see, for example, [HPS92], [GPS95], [KP00a], [GMPS10] for connections between AF algebras and minimal homeomorphisms of Cantor sets.

I.1.3.4. There has long been extensive study of crossed products, especially transformation group C*-algebras; see, for example, [Wil07]. The structure can be quite complicated and is far from fully understood. There are extensive, although incomplete, sufficient criteria for the crossed product to be simple. In particular, if a discrete amenable group G acts freely on X, i.e. no group element except the identity has a fixed point, then the transformation group C*-algebra is simple if and only if the action is *minimal*, i.e. the orbit of every point is dense in X.

I.1.3.5. Irrational Rotation Algebras. One of the most obvious examples, very important historically and in applications, is obtained by fixing an irrational number $\theta \in [0, 1]$, and letting $\mathbb{Z}$ act on the circle $\mathbb{T}$ by $n \mapsto$ (rotation by $2\pi n\theta$). The crossed product is a separable simple unital C*-algebra called the *irrational rotation algebra* with angle θ, denoted A_θ. At this point it is unclear to what extent A_θ depends on θ up to isomorphism; see ().

The C*-algebra A_θ is generated by two unitaries (I.1.6.40.) u and v which satisfy the relation $vu = e^{2\pi i\theta}uv$, and any two unitaries satisfying this relation generate a C*-algebra isomorphic to A_θ.

The irrational rotation algebras arise naturally in noncommutative differential geometry () and also in mathematical physics (the quantum Hall effect [Bel88]), as well as being natural and critical examples of simple C*-algebras to understand, and have been extensively studied by many authors (cf. I.3.6.15.). There are higher-dimensional generalizations (*noncommutative tori*) which are equally interesting and important.

I.1.3.6. Actually, there are two versions of group C*-algebras and crossed products, the "full" and the "reduced"; the reduced crossed product is a quotient of the full crossed product. This quotient map is an isomorphism if G is amenable (and sometimes more generally), so "group C*-algebra" and "crossed product" are unambiguous if G is amenable, in particular if G is abelian.

The full C*-algebra $C^*(G)$ of a (nontrivial) group G can never be simple, since it always has a one-dimensional quotient from the trivial representation of G. But certain nonamenable (discrete) groups, notably free groups on more than one generator, have simple reduced C*-algebras $C_r^*(G)$. These are the C*-analogs of the ICC groups (I.2.2.7.), although simplicity of the reduced group C*-algebra is much more restrictive than the ICC property. These simple reduced group C*-algebras are never nuclear, however.

Groupoid C*-Algebras

Quantum Groups

A more recent application of C*-algebras is to the "theory" of quantum groups. There is an interesting foundational problem here: mathematicians seem to know a quantum group when they see one, and there has been a great deal of work on quantum groups over the last several decades, some of which is widely regarded as important, first-rate mathematics (more than one Fields medal has been awarded for work in this area). But there is still as yet no generally accepted *definition* of a quantum group! (At least there is not a specific definition, only an imprecise idea that a quantum group is some type of Hopf algebra.)

C*-algebras provide a reasonable framework for making such a definition, at least a definition of a "locally compact quantum group." If G is a locally compact topological group, then $C^*(G)$ has more structure than just a C*-algebra: it has a comultiplication making it a *Hopf C*-algebra*. Similarly, the commutative C*-algebra $C_0(G)$ has a Hopf C*-algebra structure. (These are technically only Hopf algebras if G is finite: in general, the comultiplication involves C*-tensor products and multiplier algebras.) The Hopf C*-algebra $C^*(G)$ is cocommutative but not commutative (if G is nonabelian) and the Hopf C*-algebra $C_0(G)$ is commutative but not cocommutative in general. They are "dual" in a sense generalizing Pontrjagin duality. A Hopf C*-algebra which is not necessarily either commutative or cocommutative, but satisfies some additional conditions such as existence of a "Haar system," is a reasonable candidate for what a locally compact quantum group should be. See [VK74], [Wor87], [ES92], [BS93], [Wor98], [VD98], [KV00], [MNW03] for the development of the theory and [KT99], [KT00], [Kus05] for extensive surveys.

More generally, C*-algebras form a natural setting to study "quantization." See [Rie94], [Rie99]. In fact, E. Effros [Eff94] calls C*-algebras "quantized locally compact spaces," and adds:

> "One might even argue that the C*-algebras provide the only objects which have all of the algebraic and analytic characteristics of the complex numbers."

Mathematical Physics

I.1.3.7. Even before the applications to quantum groups and quantization described above, C*-algebras arose naturally in mathematical physics, particularly quantum mechanics. In fact, the first person to study operator algebras systematically was von Neumann, who was primarily motivated, at least at the beginning, by quantum mechanics; see [vN18]. Segal [Seg47] argued for the relevance of C*-algebras to physics. Specific examples of C*-algebras arising in mathematical physics are found in I.1.3.5. and I.1.7.16.. See () for an extensive survey of operator algebras in mathematical physics, and () for modern connections with quantum field theory. See [BR87], [BR97], [Con94], [Dop91], [Haa96], [Jon09]. The interaction has gone both ways: for example, modular theory for von Neumann algebras had its origin in quantum statistical mechanics (*KMS states* [HHW67]).

Topology and Geometry

Free Probability

Quantum computing

Other Applications

I.1.4. Separable C*-Algebras

I.1.4.1. For C*-algebras, separability has its usual meaning from topology: a C*-algebra is separable if it has a countable dense subset. Since a C*-algebra is metrizable, separability is equivalent to second countability.

In common with many other settings in mathematics, it is natural to restrict primary attention to separable C*-algebras, since, as may be expected, they are better behaved and far more accessible to study than general C*-algebras (see e.g. [Far19]). Thus we will for the most part restrict attention only to separable C*-algebras, although we will need to use nonseparable algebras in several constructions and arguments (infinite products, ultraproducts, multiplier and corona algebras, etc.). Most C*-algebras of general interest (other than von Neumann algebras) are separable:

> If X is a locally compact Hausdorff space, then $C_0(X)$ is separable if and only if X is second countable. If X is compact, then $C(X)$ is separable if and only if X is metrizable.
>
> If G is a locally compact topological group, then $C^*(G)$ is separable if and only if G is second countable. In particular, if G is discrete, then $C^*(G)$ is separable if and only if G is countable.
>
> If a locally compact group G acts on a C*-algebra A, then the crossed product $A \rtimes_\alpha G$ is separable if and only if A is separable and G is second countable.
>
> Since any finite-dimensional topological vector space over $\mathbb{C}$ is separable, any finite-dimensional C*-algebra is separable. In particular, $\mathbb{M}_n$ is separable.
>
> $\mathcal{K}(\mathcal{H})$ is separable if and only if $\mathcal{H}$ is separable. Thus $\mathbb{K}$ is separable.
>
> $\mathcal{B}(\mathcal{H})$ is separable if and only if $\mathcal{H}$ is finite-dimensional. Thus a concrete C*-algebra on a separable Hilbert space is not necessarily separable (cf. I.2.1.8.). But every separable C*-algebra can be isometrically represented as a concrete C*-algebra of operators on a separable Hilbert space.

I.1.4.2. If A is a C*-algebra and $S \subseteq A$, there is a smallest C*-subalgebra of A containing S, the intersection of all C*-subalgebras containing S or the closure of the *-subalgebra of A generated algebraically by S, denoted $C^*(S)$. S generates A if $C^*(S) = A$. A separable C*-algebra A is obviously countably generated (a dense subset generates A). Conversely, if A is generated by a set S, then the *-subring of A generated by scalar multiples of elements of S by scalars in $\mathbb{Q} + \mathbb{Q}i$ is dense in A, and if S is countable so is this dense subring, so A is separable. Thus a C*-algebra is separable if and only if it is countably generated.

I.1.5. Construction from Building Blocks

I.1.5.1. In the classification of finite groups, every finite group is built up from finitely many finite simple groups by means of extensions. Thus there are two steps in understanding all finite groups: first, understanding finite simple groups, and second, understanding extensions. A similar, but more complicated, principle applies also in the C*-algebra case: general C*-algebras are built up from "building blocks" which are simple C*-algebras, although the assembly is far more complex than in the finite group case. For example, commutative C*-algebras are "built up" from copies of $\mathbb{C}$.

Simple C*-Algebras

I.1.5.2. Definition. A C*-algebra A is *simple* if it has no nontrivial closed ideals.

I.1.5.3. Examples. $\mathbb{M}_n$ (including $\mathbb{C} = \mathbb{M}_1$), $\mathbb{K}$ (more generally, $\mathcal{K}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$).

I.1.5.4. There is a slight ambiguity in the definition of simplicity. We could require that A have no nontrivial ideals at all, closed or not (i.e. is simple as a ring); such an A is called *algebraically simple*. A simple unital C*-algebra is automatically algebraically simple (I.1.1.2.(viii)), so there is no ambiguity in the unital case. $\mathcal{K}(\mathcal{H})$ has many nontrivial ideals if $\mathcal{H}$ is infinite-dimensional, all of them dense; the smallest is the set of finite-rank operators (there are also the trace-class operators $\mathcal{L}^1(\mathcal{H})$, the Hilbert-Schmidt operators $\mathcal{L}^2(\mathcal{H})$, etc., cf. ()). Thus $\mathbb{K}$ is simple but not algebraically simple. There are nonunital C*-algebras which are algebraically simple, e.g. I.3.3.62.(i).

I.1.5.5. It follows easily from I.1.2.4.(ii) that the only simple commutative C*-algebra is $\mathbb{C}$. This has a more elementary proof, at least in the unital case: A unital simple commutative ring is a field, and by the Banach-Mazur theorem the only field which can be made into a complex Banach algebra is $\mathbb{C}$. In fact, commutative C*-algebras generally have lots of ideals: the closed ideals of $C_0(X)$ are in natural one-one correspondence with the closed subsets of X, the ideal corresponding to a closed subset Y being the set of functions vanishing on Y. (The nonclosed ideals of $C_0(X)$ can be extremely complicated; see e.g. [Bla06, II.5.2.1] and [GJ76].)

The Primitive Ideal Space of a C*-Algebra

I.1.5.6. It is common in mathematics, notably in algebraic geometry, to associate to a ring R a topological space made of appropriate ideals of R with the hull-kernel (Jacobsen) topology. This construction only works if the ideals are prime ideals. Using all prime ideals, we obtain the space denoted $\mathrm{Spec}(R)$; using only maximal ideals we get $\mathrm{Maxspec}(R)$.

In a C*-algebra A, it turns out that the most appropriate ideals to use are the *primitive ideals*, the kernels of irreducible representations of A. Every primitive ideal is a closed prime ideal, and conversely if A is separable every closed prime ideal is primitive, i.e. the primitive ideals are precisely the closed prime ideals. Every maximal ideal is a primitive ideal.

I.1.5.7. Definition. Let A be a C*-algebra. The *primitive ideal space* $\mathrm{Prim}(A)$ of A is the set of primitive ideals of A with the hull-kernel topology.

I.1.5.8. Recall that the hull-kernel topology is defined by closure: if $S = \{J_\lambda : \lambda \in \Lambda\}$ is a set of primitive ideals of A, the closure of S is the set of all primitive ideals of A containing $\cap_{\lambda\in\Lambda} J_\lambda$.

I.1.5.9. Here are some basic properties of $\mathrm{Prim}(A)$ for a C*-algebra A (some hold also for $\mathrm{Spec}(A)$ for a general ring A):

$\mathrm{Prim}(A)$ is T_0, but not necessarily T_1.

If A is unital, $\mathrm{Prim}(A)$ is compact. If A is nonunital, then $\mathrm{Prim}(A)$ may or may not be compact. $\mathrm{Prim}(A)$ is always "locally compact" in the sense that every point of $\mathrm{Prim}(A)$ has a neighborhood base of compact sets, although a point need not have any closed compact neighborhood. (There are several inequivalent potential definitions of local compactness in the non-Hausdorff case, none of which is generally accepted as *the* definition by topologists. See [Bla, XI.11.8.20].)

$\mathrm{Prim}(A)$ is a Baire space ().

If A is separable, then $\mathrm{Prim}(A)$ is second countable.

I.1.5.10. Examples. (i) Every primitive ideal of $C_0(X)$ is maximal, and consists of the set M_p of functions vanishing at some fixed point p of X. Thus $\mathrm{Prim}(C_0(X))$ can be naturally identified (as a set) with X. It is a good exercise (cf. ()) to show that this identification is a homeomorphism. This is one way the space X can be recovered from $C_0(X)$ (I.1.2.4.(ii); see I.1.6.6.(i) for another way). In particular, $\mathrm{Prim}(C_0(X))$ is locally compact and Hausdorff.

(ii) Let A be the C*-subalgebra of $C([0,1],\mathbb{M}_2)$ consisting of all functions for which $f(1)$ is a diagonal matrix. For $t \in [0,1)$, there is a maximal ideal M_t consisting of all functions vanishing at t. There are two maximal ideals corresponding to $t = 1$:

$$M_{11} = \left\{ f \in A : f(1) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \text{ for some } x \in \mathbb{C} \right\}$$

$$M_{12} = \left\{ f \in A : f(1) = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \text{ for some } x \in \mathbb{C} \right\}$$

and these are all the primitive ideals. Thus $\mathrm{Prim}(A)$ is $[0,1]$ with the point 1 "doubled." The topology is the usual topology on $[0,1)$, and if $t_n \to 1$, the corresponding sequence (M_{t_n}) in $\mathrm{Prim}(A)$ converges simultaneously to both M_{11} and M_{12}. Thus $\mathrm{Prim}(A)$ is compact and T_1, but not Hausdorff.

The nonunital C*-subalgebras M_{11} and M_{12} (which are isomorphic, in fact conjugate under an automorphism of A) have $\mathrm{Prim}(M_{11}) \cong [0,1]$ with the usual topology. Similarly, if B is the (unital) C*-subalgebra of A consisting of functions f with $f(1)$ a scalar multiple of the identity, then $\mathrm{Prim}(B) \cong [0,1]$ with the usual topology.

(iii) Let $A = \mathbb{K} + \mathbb{C}1$ as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ separable and infinite-dimensional). Then A has exactly two primitive ideals, 0 and $\mathbb{K}$ (in fact these are the only two closed proper ideals). $\{0\}$ is a dense open set in $\mathrm{Prim}(A)$, and $\{\mathbb{K}\}$ is closed. Thus $\mathrm{Prim}(A)$ is T_0 but not T_1. The same is true for $\mathcal{B}(\mathcal{H})$ itself.

I.1.5.11. Every closed ideal in a C*-algebra is an intersection of primitive ideals. Thus the closed ideals of a C*-algebra A are in one-one correspondence with the closed sets in $\mathrm{Prim}(A)$. In particular, $\mathrm{Prim}(A)$ is a one-point space if and only if A is simple.

If J is a closed ideal in A, then we can identify $\mathrm{Prim}(A/J)$ with

$$\mathrm{Prim}_J(A) = \{K \in \mathrm{Prim}(A) : J \subseteq K\},$$

the corresponding closed set in $\mathrm{Prim}(A)$. There is a map ρ_J from the complementary open set

$$\mathrm{Prim}^J(A) = \{K \in \mathrm{Prim}(A) : J \not\subseteq K\}$$

to $\mathrm{Prim}(J)$, defined by $\rho_J(K) = K \cap J$, which is a homeomorphism ().

Some Philosophy

I.1.5.12. It is an oversimplification, but good intuition, that a general C*-algebra A has a simple C*-algebra sitting at each primitive ideal (this is literally true at maximal ideals, the simple C*-algebra at the maximal ideal M being A/M, and works also at some, but not all, nonmaximal primitive ideals), and that A is built up from these simple C*-algebras by some complicated assembly process. We will have little more to say about this except to take it as motivation that to understand general C*-algebras, a reasonable (and necessary) first step is to understand simple C*-algebras.

I.1.5.13. Under this philosophy, the Type I or GCR C*-algebras are precisely the ones built up from *elementary C*-algebras*, (simple) C*-algebras of the form $\mathcal{K}(\mathcal{H})$ for various Hilbert spaces $\mathcal{H}$, or in the separable case exactly finite-dimensional matrix algebras and $\mathbb{K}$ (they are exactly the Type I simple C*-algebras). To go beyond this class, we need a larger supply of simple building blocks.

Inductive Limits

If we want more simple C*-algebras, how will we get them? Crossed products are a source, but they are rather hard to analyze. But there is a much simpler construction, the inductive limit, which gives an entirely different way of building a C*-algebra from well-understood building blocks. Instead of assembling C*-algebras from simple C*-algebras, we build them up from subalgebras; we can actually turn the process around and make simple C*-algebras from nonsimple ones.

I.1.5.14. The setup is that we have a sequence (A_n) of C*-algebras and an embedding $\phi_{n,n+1}$ of A_n as a C*-subalgebra of A_{n+1} for each n (the construction can be done more generally over any directed index set, but we stick with $\mathbb{N}$ which is almost exclusively used in the classification program).

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots$$

We can form the "union" or *algebraic direct limit* A_∞ of the A_n (the construction can be done in such a way that this is an honest union, e.g. in the quotient algebra $[\prod A_n]/[\oplus A_n]$). Then A_∞ has a natural induced *-algebra structure and a C*-norm (note that all the embeddings are isometries).

Important Technicality: A_∞ is not complete except in trivial cases. Thus we complete it to make it a C*-algebra.

I.1.5.15. Definition. The completion of A_∞ is the *inductive limit* of the inductive system $(A_n, \phi_{n,n+1})$, denoted $\varinjlim(A_n, \phi_{n,n+1})$.

I.1.5.16. There is a natural embedding ϕ_n from A_n into the inductive limit, which is consistent with the embeddings of each A_n into the next (there are also consistent embeddings $\phi_{n,m}$ of A_n into A_m for any $n < m$, defined by composition). Note that if each A_n is unital and the $\phi_{n,n+1}$ are unital, then the inductive limit is also unital and the ϕ_n are unital.

I.1.5.17. The completion causes some brief headaches in the analysis of the inductive limit, but turns out to have almost no bad effects. In fact, far from being a negative, it "smooths out" the algebraic direct limit A_∞ in such a way that the completed inductive limits of quite different inductive systems can be isomorphic, and no classification theorem would be possible without this smoothing (cf. I.3.5.8.).

As an example of how things actually behave well, if A is the inductive limit, then A_∞ is a dense *-subalgebra of A. If J is a closed ideal of A, we would not expect the ideal $J \cap A_\infty$ of A_∞ to necessarily be very large; it could conceivably even be 0 for nonzero J. In fact, this can happen for a general dense *-subalgebra and closed ideal in a C*-algebra. But in the inductive limit case, the situation is very pretty:

I.1.5.18. Proposition. Let $(A_n, \phi_{n,n+1})$ be an inductive system of C*-algebras, A_∞ the algebraic direct limit, and A the inductive limit. If J is any closed ideal of A, then $J \cap A_\infty$ is dense in J. In other words, if $J_n = J \cap \phi_n(A_n)$, then $\cup_n J_n$ is dense in J.

This is a nice consequence of the uniqueness of norm on a C*-algebra (II.1.12.3.).

I.1.5.19. Thus the closed ideals of A correspond to sequences (J_n), where J_n is a closed ideal of A_n and $\phi_{n,n+1}(J_n) = J_{n+1} \cap \phi_{n,n+1}(A_n)$ (coherent sequences of ideals). This gives a convenient and usable criterion for telling when the inductive limit is simple, which can be satisfied rather easily (cf. I.3.2.9.). As a (very) special case, if each A_n is simple, then the inductive limit is simple.

I.1.5.20. The inductive limit construction can be done more generally if the $\phi_{n,n+1}$ are just homomorphisms, with scarcely any modification (the only technicality that causes problems is that the inductive limit can be the zero C*-algebra if the A_n or ϕ_n are nonunital). The only difference is that the ϕ_n are then only homomorphisms into the inductive limit. We will primarily consider only inductive systems with injective connecting maps, which are easier to visualize.

I.1.5.21. In our construction of simple C*-algebras, we will successively consider inductive limits where the A_n are matrix algebras, finite-dimensional C*-algebras (finite direct sums of matrix algebras), and locally homogeneous and subhomogeneous C*-algebras (subalgebras of direct sums of matrix algebras over commutative C*-algebras). It turns out that an unexpectedly large but well-behaved class of simple C*-algebras can be constructed in this manner, and since the building blocks are easily analyzable, so are these inductive limits (but the analysis is not so easy!).

I.1.6. Traces, Projections, Unitaries, and Equivalence

The key to understanding the structure of simple C*-algebras, at least the ones we are considering, is to examine the equivalence classes of projections and unitaries, and as a means to this and an additional invariant in the absence of enough projections, the traces on the algebra. In fact, these invariants, which together comprise the *Elliott invariant*, turn out to tell the whole story.

Traces

I.1.6.1. Definition. Let A be a unital C*-algebra. A *state* on A is a linear functional σ with $\sigma(x^*x) \geq 0$ for all $x \in A$ and $\sigma(1) = 1$. Let $\mathcal{S}(A)$ denote the set of states on A. Then $\mathcal{S}(A)$ is a compact convex subset of the dual space A^* of A with the weak-* topology. An extreme point of $\mathcal{S}(A)$ is called a *pure state* of A.

A *trace* on A is a state τ on A with $\tau(xy) = \tau(yx)$ for all $x, y \in A$. Let $\mathcal{T}(A)$ denote the set of traces on A. $\mathcal{T}(A)$ is a compact convex subset of $\mathcal{S}(A)$.

I.1.6.2. A state automatically has norm 1. What we have called a "trace" should be called a "tracial state" or "normalized trace" but we will just call them traces for economy of terminology. Nonunital C*-algebras, and nonsimple unital C*-algebras, also sometimes have *unbounded traces* (I.1.6.9.), but a simple unital C*-algebra cannot have an unbounded trace. All traces considered in this book will be finite unless otherwise qualified, and almost always normalized, but we will sometimes need to consider *unnormalized traces*, e.g. when we pass to matrix algebras (I.1.6.8.).

I.1.6.3. If A is a concrete C*-algebra of operators acting nondegenerately on $\mathcal{H}$ and $\xi \in \mathcal{H}$, and $\phi_\xi(x) = \langle x\xi, \xi \rangle$ for $x \in A$, then ϕ_ξ is a positive linear functional on A of norm $\|\xi\|^2$, so ϕ_ξ is a state if $\|\xi\| = 1$. Such a state is called a *vector state* of A. This example is the origin of the term "state": in the mathematical formulation of quantum mechanics, the states of a physical system are given by probability distributions (unit vectors in an L^2-space), and observables are self-adjoint operators; the value of the observable T on the state ξ is $\langle T\xi, \xi \rangle$ (cf. [vN18], [BR87]).

I.1.6.4. If σ is a state on a C*-algebra A, there is a construction, called the *GNS construction* [Bla06, II.6.4], of a representation π_σ of A. The representation π_σ is irreducible if (and only if) σ is a pure state. The state σ is a vector state from π_σ.

I.1.6.5. Every C*-algebra has many states, in fact $\mathcal{S}(A)$ spans A^*. A stronger result holds: if $x \in A$, then there is a pure state σ on A with $\sigma(x^*x) = \|x\|^2$ (this fact, which is an easy consequence of the Hahn-Banach and Krein-Milman theorems, is the key step in the proof of Theorem I.1.2.4.(i), using I.1.6.4.). However, a simple unital C*-algebra need not have any traces, i.e. $\mathcal{T}(A)$ can be empty (). When $\mathcal{T}(A)$ is nonempty, it is always a Choquet simplex (I.3.3.55.) if A is unital (if A is nonunital, then the set $\mathcal{T}_0(A)$ of traces of norm ≤ 1 is a Choquet simplex, isomorphic to $\mathcal{T}(A^\dagger)$). If A is separable, then both $\mathcal{S}(A)$ and $\mathcal{T}(A)$ are metrizable in the weak-* topology.

I.1.6.6. Examples. (i) Any state on a commutative C*-algebra is a trace. A state on $C(X)$ is given by integration from a Radon probability measure on X (just a Borel probability measure on X if X is metrizable). The pure states correspond to point masses, and thus are multiplicative, i.e. homomorphisms to $\mathbb{C}$ (*characters*). The kernel of such a character is a maximal ideal, hence corresponds to a point of X. The weak-* topology, or topology of pointwise convergence, coincides with the hull-kernel topology on $\mathrm{Prim}(C(X))$ and the original topology on X. This is another way to recover X from $C(X)$ (I.1.2.4.(ii), I.3.3.53.). $\mathcal{S}(C(X)) = \mathcal{T}(C(X))$ is a *Bauer simplex*, a Choquet simplex whose set of extreme points is closed (and homeomorphic to X).

(ii) As shown in linear algebra, there is a unique trace on $\mathbb{M}_n$, whose value on a matrix is the sum of the diagonal elements. However, our requirement that $\tau(1) = 1$ means we have to divide the standard trace by n, so the unique trace τ in our sense is given on a matrix $a = (a_{ij})$ by

$$\tau(a) = \frac{1}{n}\sum_{j=1}^{n} a_{jj}$$

which can be profitably interpreted as the average of the diagonal elements of the matrix.

I.1.6.7. If σ is a state on a C*-algebra A, then $N_\sigma = \{x \in A : \sigma(x^*x) = 0\}$ is a closed left ideal of A (since $x^*y^*yx \leq \|y\|^2 x^*x$ and hence $0 \leq \sigma(x^*y^*yx) \leq \|y\|^2\sigma(x^*x)$ for any $x, y \in A$); if τ is a trace on A, then N_τ is a closed two-sided ideal since $\tau(y^*x^*xy) = \tau(xyy^*x^*) \leq \|y\|^2\tau(xx^*) = \|y\|^2\tau(x^*x)$ for $x, y \in A$. Thus if A is a simple C*-algebra, then any trace τ on A is *faithful*, i.e. $\tau(x^*x) > 0$ if $x \neq 0$.

I.1.6.8. If τ is a trace on a C*-algebra A, then τ defines an unnormalized trace on $M_n(A)$ for any n by the formula

$$\tau_n((a_{ij})) = \sum_{j=1}^{n} \tau(a_{jj})$$

(unnormalized since $\tau_n(1_{M_n(A)}) = n$), and a trace on $M_n(A)$ by the formula

$$\tau((a_{ij})) = \frac{1}{n}\sum_{j=1}^{n} \tau(a_{jj})$$

(this can be regarded as extending τ if A is regarded as a diagonal subalgebra of $M_n(A)$). Conversely, any trace on $M_n(A)$ comes from a trace on A in the same way. Thus there is a natural isomorphism (affine homeomorphism) between $\mathcal{T}(M_n(A))$ and $\mathcal{T}(A)$.

More generally, if A is a unital C*-subalgebra of a C*-algebra B, any trace on B restricted to A is a trace on A, although not every trace on A can be obtained in this way in general; thus there is a continuous affine map from $\mathcal{T}(B)$ to $\mathcal{T}(A)$ given by restriction (which is neither injective nor surjective in general). Even more generally, if $\phi : A \to B$ is a unital homomorphism, then $\tau \mapsto \tau \circ \phi$ gives a continuous affine map from $\mathcal{T}(B)$ to $\mathcal{T}(A)$, i.e. $A \mapsto \mathcal{T}(A)$ is contravariantly functorial (in the unital category).

Unbounded Traces

In the nonunital case we will also have to consider unbounded traces. We make the following definition:

I.1.6.9. Definition. Let A be a C*-algebra. An *lsc trace* on A is a function $\tau : A_+ \to [0, +\infty]$ satisfying:

(i) $\tau(0) = 0$.

(ii) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in A_+$.

(iii) $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in A_+$, $\lambda \in (0, +\infty)$.

(iv) $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$.

(v) τ is lower semicontinuous.

Denote by $\bar{\mathcal{T}}(A)$ the set of lsc traces on A.

I.1.6.10. A trace in the sense of I.1.6.1., restricted to the positive elements of A, is an lsc trace. If τ is an lsc trace and $\alpha \in (0, +\infty)$, then $\alpha\tau$ is an lsc trace; thus $\bar{\mathcal{T}}(A)$ is a cone. (Multiplication of an lsc trace by 0 or $+\infty$ is a little more subtle to define carefully.) If $0 \leq a \leq b$ in A and τ is an lsc trace on A, then

$$\tau(a) \leq \tau(a) + \tau(b - a) = \tau(b)$$

so τ is order-preserving. In particular, if A is unital and $\tau(1_A) < \infty$, then τ takes only finite values on A_+.

I.1.6.11. Any C*-algebra A has two "extremal" lsc traces: the zero trace and the "infinite" trace τ_∞ defined by $\tau_\infty(0) = 0$ and $\tau_\infty(a) = +\infty$ if $0 \neq a \in A_+$. Some C*-algebras have no others. Denote by $\tilde{\mathcal{T}}(A)$ the set of lsc traces with the two extremal ones removed. Then $\tilde{\mathcal{T}}(A)$ is also a cone (possibly empty).

I.1.6.12. By I.1.6.7., if τ is an lsc trace on A, then N_τ is an ideal of A, which is closed since τ is lower semicontinuous. Thus a nonzero lsc trace on a simple C*-algebra is faithful. By a similar argument, the set $\{x \in A : \tau(x^*x) < +\infty\}$ is an ideal of A, not closed in general. Thus, if A is simple and τ is not τ_∞, this ideal is dense in A. So τ is finite on the positive cone of the Pedersen ideal $Ped(A)$ (), and extends to a (positive) linear functional on $Ped(A)$. In particular, if A is simple and unital, then any lsc trace on A other than τ_∞ comes from an (unnormalized) trace on A. (The situation is far more complicated in general for nonsimple C*-algebras.) We also have:

I.1.6.13. PROPOSITION. Let A be a C*-algebra and τ an lsc trace on A. The following are equivalent:

(i) τ takes only finite values on A_+.

(ii) τ is bounded in the sense that there is a constant K such that $\tau(a) \leq K\|a\|$ for all $a \in A_+$.

(iii) τ is continuous.

PROOF: (ii) $\Rightarrow$ (i) is trivial. For (i) $\Rightarrow$ (ii), suppose τ is not bounded. Then there is a sequence (a_n) in A_+ with $\|a_n\| = 1$ for all n and $\tau(a_n) \geq 4^n$. If we set

$$a = \sum_{n=1}^{\infty} 2^{-n} a_n$$

in A, we have for all n that $a \geq 2^{-n}a_n$, so $\tau(a) \geq 2^{-n}\tau(a_n) \geq 2^n$ and thus $\tau(a) = +\infty$.
(ii) $\iff$ (iii) in the presence of (i) (so τ extends to a linear functional on A) is a standard fact in Banach spaces, and hence (i) $\Rightarrow$ (iii). For (iii) $\Rightarrow$ (ii) in general, if τ is unbounded, let (a_n) be as above; then $2^{-n}a_n \to 0$ but $\tau(2^{-n}a_n) \not\to \tau(0) = 0$, so τ cannot be continuous.

I.1.6.14. Thus, if A is a simple C*-algebra, the lsc traces in $\tilde{\mathcal{T}}(A)$ may be either bounded or unbounded; the bounded ones are just scalar multiples of traces (tracial states). There may be some of each. If A is unital or, more generally, algebraically simple, then all lsc traces on A except τ_∞ are bounded. At the other extreme, if A is stable, all nonzero lsc traces on A are unbounded.

Some references use the term *unbounded trace* to mean what we call an lsc trace, sometimes without the lower semicontinuity restriction (even simple C*-algebras can have unbounded traces which are not lower semicontinuous, which can be very pathological, e.g. with dense N_τ, although some like the Dixmier trace () on $\mathbb{K}$ are important in other contexts). We prefer our terminology partly because of the lower semicontinuity ambiguity, and partly because it is terminologically absurd to allow an unbounded trace to be bounded.

I.1.6.15. It is a somewhat nontrivial fact (cf. I.1.6.8.) that any lsc trace on a C*-algebra A extends uniquely to an lsc trace on $M_n(A)$ for any n and a (necessarily unbounded) lsc trace on $A \otimes \mathbb{K}$. It is immediate that an lsc trace τ on A restricts to an lsc trace on any C*-subalgebra B; if A is simple and B is a hereditary C*-subalgebra, and τ is not τ_∞ on A (i.e. it takes a finite nonzero value), the same is true for the restricted lsc trace. Thus, if A and B are stably isomorphic simple C*-algebras, there is a natural affine isomorphism between $\tilde{\mathcal{T}}(A)$ and $\tilde{\mathcal{T}}(B)$. (This isomorphism does not preserve boundedness in general.)

Projections

I.1.6.16. Definition. Let A be a C*-algebra. A *projection* in A is a self-adjoint idempotent, i.e. an element p such that $p = p^* = p^2$.

I.1.6.17. A has at least one projection, 0; if A is unital, 1 is also a projection. It need not have any others (*nontrivial projections*). A nonzero projection p has norm 1 by the C*-axiom, and is positive since $p = p^*p$. In fact, by functional calculus, an element p of a C*-algebra is a projection if and only if $p = p^*$ and $\sigma(p) \subseteq \{0, 1\}$ ($\sigma(p) = \{0, 1\}$ if p is nontrivial).

I.1.6.18. Examples. (i) A projection in $\mathcal{B}(\mathcal{H})$ is just a projection operator in the usual sense, the orthogonal projection onto a closed subspace. Since there are many closed subspaces (unless $\mathcal{H}$ is one-dimensional), there are many projections in $\mathcal{B}(\mathcal{H})$. The projections in a C*-algebra are thus exactly the elements which become projection operators when the C*-algebra is represented as a concrete C*-algebra of operators.

(ii) A projection in $C_0(X)$ is just the characteristic or indicator function of a compact open subset of X. Thus if X is connected, there are no nontrivial projections in $C_0(X)$.

I.1.6.19. The partial ordering on a C*-algebra A from the positive cone gives a partial ordering on the set $\mathrm{Proj}(A)$ of projections of A. If $p, q \in \mathrm{Proj}(A)$, the following are equivalent (cf. II.1.8.1.):

(i) $p \leq q$.

(ii) $p \leq \lambda q$ for some $\lambda > 0$.

(iii) $pq = p$ (which implies that $qp = (pq)^* = p^* = p$ also).

(iv) $q - p$ is a projection.

If these conditions are satisfied, p is a *subprojection* of q. In particular, if A is unital, then any projection p is a subprojection of 1, i.e. $0 \leq p \leq 1$.

I.1.6.20. Two projections p and q are *orthogonal*, written $p \perp q$, if $pq = 0$ (thus also $qp = (pq)^* = 0$). If $p \perp q$, then $p + q$ is a projection (the converse holds also). If A is unital, then any projection p has an *orthogonal complement* $1 - p$; a projection q in A satisfies $q \perp p$ if and only if $q \leq 1 - p$.

Isometries and Partial Isometries

I.1.6.21. Definition. Let A be a C*-algebra. An element $x \in A$ is a *partial isometry* if x^*x is a projection.

Let A be a unital C*-algebra. An element $x \in A$ is an *isometry* if $x^*x = 1$.

I.1.6.22. An isometry is a partial isometry. A projection is a partial isometry. A nonzero partial isometry has norm 1 by the C*-axiom. If x is a partial isometry, then xx^* is also a projection (i.e. x^* is also a partial isometry). In fact, x is a partial isometry if and only if $x = xx^*x$ [if x is a partial isometry, set $y = x - xx^*x$; then $y^*y = 0$, so $y = 0$.] In particular, if x is an isometry, then xx^* is a projection.

The name "isometry" is appropriate since if x is an isometry in a unital C*-algebra A, then for any $y \in A$ we have

$$\|xy\|^2 = \|y^*x^*xy\| = \|y^*y\| = \|y\|^2$$

so $\|xy\| = \|y\|$.

I.1.6.23. Example. If T is a partial isometry in $\mathcal{B}(\mathcal{H})$, then there is a closed subspace $\mathcal{X}$ of $\mathcal{H}$ such that T maps $\mathcal{X}$ isometrically onto another closed subspace $\mathcal{Y}$ of $\mathcal{H}$ and is zero on $\mathcal{X}^\perp$. Then T^*T is the orthogonal projection onto $\mathcal{X}$, and TT^* the orthogonal projection onto $\mathcal{Y}$. In particular, if T is an isometry, then $\mathcal{X} = \mathcal{H}$, so T maps $\mathcal{H}$ isometrically onto the subspace $\mathcal{Y}$ (which may or may not be all of $\mathcal{H}$ if $\mathcal{H}$ is infinite-dimensional). This is the origin of the names "partial isometry" and "isometry"; [partial] isometries in a general C*-algebra are elements which become [partial] isometry operators when the algebra is represented on a Hilbert space.

Equivalence of Projections

I.1.6.24. Definition. If p and q are projections in A, then p and q are *equivalent (in A)*, written $p \sim q$, if there is a partial isometry $x \in A$ with $x^*x = p$, $xx^* = q$ (x is called a *partial isometry from p to q*), and $p \precsim q$ if p is equivalent to a subprojection of q, i.e. if there is a partial isometry $x \in A$ with $x^*x = p$ and $xx^* \leq q$ (we say p is *subordinate* to q).

Although the notation does not reflect it (we could write $\sim_A$ if necessary to specify), equivalence of projections is relative to a specified containing C*-algebra. This equivalence is properly called *Murray-von Neumann equivalence*, but we will just call it equivalence. It is indeed an equivalence relation, since if x is a partial isometry from p to q, then $x = xp = qx = qxp$, so if $x^*x = p$, $xx^* = y^*y = q$, and $yy^* = r$, then $(yx)^*(yx) = p$ and $(yx)(yx)^* = r$; and x^* is a partial isometry from q to p (and p itself is a partial isometry from p to p).

In a commutative C*-algebra, equivalence is trivial: $p \sim q$ if and only if $p = q$, and $p \precsim q$ if and only if $p \leq q$.

I.1.6.25. The relation $\precsim$ is transitive and is really defined on equivalence classes: if $p \sim p'$ and $q \sim q'$, then $p \precsim q$ if and only if $p' \precsim q'$. Note that $p \precsim q$ and $q \precsim p$ do not together imply $p \sim q$ in general, e.g. in O_n, $n > 2$ (I.5.1.16.), i.e. there is no Schröder-Bernstein theorem for equivalence of projections, although there is in the finite case (I.1.6.39.).

I.1.6.26. Example. Two projections in $\mathcal{B}(\mathcal{H})$ are equivalent if and only if their ranges have the same (orthogonal) dimension. Thus, if $\mathcal{H}$ is separable and infinite-dimensional, all projections in $\mathcal{B}(\mathcal{H})$ with infinite-dimensional range are equivalent. Equivalence classes of finite-rank projections are parametrized by the dimension of the range, which is a nonnegative integer, i.e. the equivalence classes of projections in $\mathcal{B}(\mathcal{H})$ are parametrized by $\{0\} \cup \mathbb{N} \cup \{\infty\}$. Subordination is given by the usual order on this set.

There are many projections in $\mathbb{M}_n$ (if $n > 1$), but precisely $n+1$ equivalence classes, parametrized by $\{0, 1, \ldots, n\}$. The equivalence classes of 0 and 1 (parametrized by n) are singletons, and the others are fairly large.

I.1.6.27. It is useful in general to regard two projections in a C*-algebra to be equivalent if and only if they have the same "size" (or, more precisely, their supports have the same "size" but possibly different "location.")

I.1.6.28. If $p \sim q$, $p' \sim q'$, $p \perp p'$, $q \perp q'$, then $p + p' \sim q + q'$: if $x^*x = p$, $xx^* = q$, $y^*y = p'$, $yy^* = q'$, then $(x+y)^*(x+y) = p + p'$, $(x+y)(x+y)^* = q + q'$. The same is true if $\sim$ is replaced by $\precsim$. In particular, if $p \sim q$ and r is a projection with $p \perp r$, $q \perp r$, then $p + r \sim q + r$.

I.1.6.29. One significant technical fact, a straightforward consequence of I.1.1.2.(vii), is that if p and q are projections in a C*-algebra A, and $\|p - q\| < 1$, then $p \sim q$ (II.1.8.4.). This implies that a separable C*-algebra can have only countably many equivalence classes of projections.

Traces and Equivalence

I.1.6.30. If A is a C*-algebra and σ is a state on A, then $\sigma(p) \geq 0$ (in fact $0 \leq \sigma(p) \leq 1$) for any projection p in A. If A is a C*-algebra and τ a (bounded or unbounded) trace on A, and p and q are equivalent projections in A with x a partial isometry from p to q, then

$$\tau(p) = \tau(x^*x) = \tau(xx^*) = \tau(q)$$

so τ is constant on equivalence classes. (The converse is false in general: nonequivalent projections can have the same trace, even in a simple unital C*-algebra with unique trace (I.3.2.10.).) If $p \precsim q$, then $\tau(p) \leq \tau(q)$; again the converse does not hold in general.

I.1.6.31. Example. If τ is the unique (normalized) trace on $\mathbb{M}_n$ and p is a rank k projection in $\mathbb{M}_n$, then $\tau(p) = \frac{k}{n}$. Thus the values τ takes on projections are exactly

$$\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

and these parametrize the equivalence classes, i.e. $p \sim q$ if and only if $\tau(p) = \tau(q)$, and $p \precsim q$ if and only if $\tau(p) \leq \tau(q)$.

Finite and Infinite Projections

The next definition is reminiscent of the definition of an infinite (Dedekind-infinite) set as a set equipotent with a proper subset of itself (since von Neumann's first work was in set theory, the analogy may not be farfetched).

I.1.6.32. Definition. Let A be a C*-algebra. A projection p is *infinite* (in A) if there is a projection $q \in A$ with $q \leq p$, $q \neq p$, $q \sim p$ (in A). A projection p is *finite* (in A) if it is not infinite, i.e. if $p \sim q \leq p$ implies $q = p$. The projection p is *properly infinite* (in A) if there are projections $q_1, q_2 \in A$ with $q_1, q_2 \leq p$, $q_1 \perp q_2$, and $q_1 \sim q_2 \sim p$.

I.1.6.33. A nonzero properly infinite projection is infinite (it is convenient to regard 0 as both finite and properly infinite). Any subprojection of a finite projection is finite. Note that the property of being finite or [properly] infinite is very much dependent on the ambient algebra: if A is a C*-subalgebra of B, a projection $p \in A$ which is finite in A can be infinite or even properly infinite in B (but if it is finite in B, it must be finite in A).

I.1.6.34. Definition. Let A be a unital C*-algebra. A is *finite* [resp. *infinite, properly infinite*] if 1_A is a finite [resp. infinite, properly infinite] projection in A. A is *stably finite* if $M_n(A)$ is finite for all n.

All projections in a finite C*-algebra are finite; thus a (unital) C*-algebra is infinite if and only if it contains an infinite projection.

I.1.6.35. Example. An infinite projection need not be properly infinite, and an infinite C*-algebra need not be properly infinite. Let s be the *unilateral shift* on ℓ^2:

$$s(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$$

and let T be the C*-subalgebra of $\mathcal{B}(\ell^2)$ generated by s. T is called the *Toeplitz algebra*. T is infinite since ss^* is a proper projection, onto the subspace of sequences with first coordinate zero. The projection $q = 1 - ss^*$ has rank one. It is easy to see that T contains the standard matrix units corresponding to the canonical orthonormal basis of ℓ^2, and hence T contains $\mathcal{K}(\ell^2) \cong \mathbb{K}$. Modulo $\mathbb{K}$, $ss^* = 1$, so it follows that $T/\mathbb{K} \cong C(\mathbb{T})$. Every projection in T equivalent to 1 is of the form $1 - p$ for a projection $p \in \mathbb{K}$ (and every such projection is equivalent to 1). Thus T is not properly infinite (I call such a C*-algebra "poorly infinite").

I.1.6.36. However, a projection in a simple C*-algebra is either finite or properly infinite (III.2.1.2.), and in particular a simple unital C*-algebra is either finite or properly infinite.

I.1.6.37. If A has a faithful trace τ, and $x \in A$, $x^*x = 1$, and $xx^* = q$, then $\tau(q) = \tau(1) = 1$, so $\tau(1-q) = 0$ and thus $1 - q = 0$, $q = 1$. Thus a (unital) C*-algebra with a faithful trace must be finite. Since a faithful trace on A extends to a faithful trace on $M_n(A)$ for any n, a (unital) C*-algebra with a faithful trace is stably finite. In particular, since any trace on a simple C*-algebra is faithful (I.1.6.7.), a simple unital C*-algebra with a trace is stably finite.

I.1.6.38. The converse is much harder: must a stably finite simple unital C*-algebra have a trace? This is still unknown, and is the oldest open problem in operator algebras, dating (in disguised form) to [MVN36]. It is known [Han81] that such a C*-algebra must have a tracelike function called a *quasitrace*, and by a remarkable result of Haagerup [HT99], every quasitrace on an exact (e.g. nuclear) C*-algebra is a trace (see () for a simple proof in the case of finite nuclear dimension from [BW11]); thus it is known that a simple unital nuclear C*-algebra has a trace if and only if it is stably finite.

The Schröder-Bernstein Theorem for equivalence does hold in finite C*-algebras:

I.1.6.39. Proposition. Let p and q be projections in a C*-algebra A. If $p \precsim q$ and $q \precsim p$ and p is finite, then $p \sim q$ (so q is also finite).

Proof: Let $x \in A$ with $x^*x = p$ and $xx^* = p' \leq q$, and $y \in A$ with $y^*y = q$, $yy^* = q' \leq p$. Set $p'' = yp'y^*$; then p'' is a projection with $p'' \leq q' \leq p$, and $y^*p''y = p'$. Set $z = p''yx$. Then $z^*z = x^*(y^*p''y)x = x^*p'x = p$ and $zz^* \leq p''$ (actually it is easily seen that $zz^* = p''$). So $zz^* = p$ by finiteness of p; since $zz^* \leq p'' \leq q' \leq p$, we have $p'' = q'$ and hence $p' = y^*p''y = y^*q'y = q$ and $q \sim p$. 👍

Unitaries

Although unitaries will not come explicitly into the classification work until later, they are basic objects in (unital) C*-algebras, perhaps even more basic than projections.

I.1.6.40. Definition. Let A be a unital C*-algebra. An element $u \in A$ is *unitary* if $u^*u = uu^* = 1$. Let $\mathcal{U}(A)$ denote the set of unitaries in A.

Thus u is unitary if and only if u is invertible and $u^{-1} = u^*$, if and only if both u and u^* are isometries.

I.1.6.41. A product of two unitaries is a unitary. If A is a unital C*-algebra, then $\mathcal{U}(A)$ is a group (in fact, a topological group) under multiplication.

I.1.6.42. Examples. (i) A unitary in $\mathcal{B}(\mathcal{H})$ is an ordinary unitary operator, an isometric linear map from $\mathcal{H}$ onto $\mathcal{H}$. In particular, a unitary element of $\mathbb{M}_n$ is a unitary matrix in the usual sense. The unitary elements of $\mathbb{C}$ are the complex numbers of absolute value 1, i.e. $\mathcal{U}(\mathbb{C}) = \mathbb{T}$.

(ii) A unitary element of $C(X)$ is a function taking values in $\mathbb{T}$, i.e. a continuous function from X to $\mathbb{T}$.

I.1.6.43. A unital C*-algebra has many unitaries: every element is a linear combination of four unitaries, and in fact, the convex hull of $\mathcal{U}(A)$ contains the open unit ball of A [Bla06, II.3.2.15]. Unitaries are extreme points of the closed unit ball of A, and are frequently (e.g. in a finite simple unital C*-algebra) precisely the extreme points [Bla06, II.3.2.17].

I.1.6.44. If u is a unitary in A and $\|u - 1\| < 2$, then there is a logarithm x for u, an element satisfying $x = x^*$, $\|x\| < \pi$, with $u = e^{ix}$ (). The path $u_t = e^{itx}$ is a path of unitaries from 1 to u in A. More generally, if u and v are unitaries and $\|u - v\| < 2$, then there is a path of unitaries from u to v. The connected component of 1 in $\mathcal{U}(A)$ is an open subgroup, and consists precisely of all finite products of exponentials of the form e^{ix}, $x = x^*$. (Whether there is a unitary path of controlled length between unitaries in the same component is a subtle question which has played a role in the classification program ().)

I.1.6.45. If A is a unital C*-algebra and u is a unitary in A, then $x \mapsto u^*xu$ is an automorphism of A, called an *inner automorphism*. If B is another unital C*-algebra and ϕ, ψ are (unital) homomorphisms from A to B, then ϕ and ψ are *unitarily equivalent* if there is a unitary $u \in B$ with $\psi(x) = u^*\phi(x)u$ for all $x \in A$.

Polar Decomposition

Operators in $\mathcal{B}(\mathcal{H})$ have a polar decomposition (), but general elements of a (unital) C*-algebra do not. However, invertible elements do:

I.1.6.46. Proposition. Let A be a unital C*-algebra, and x an invertible element of A. Then there are a unique unitary u and a unique positive element a (necessarily invertible) such that $x = ua$. The u and a are continuous functions of x.

Proof: Since x is invertible, so is x^*, and hence x^*x is invertible. If $a = (x^*x)^{1/2}$ is the unique positive square root of x^*x, then $a(x^*x)^{-1}$ is a right inverse for a and $(x^*x)^{-1}a$ is a left inverse for a, and thus a is invertible (and the left and right inverses are equal). Write $(x^*x)^{-1/2}$ for a^{-1}, and set $u = x(x^*x)^{-1/2}$. Then u is invertible, and $u^*u = 1$, i.e. u is unitary, and clearly $x = ua$. For uniqueness, if x is also equal to vb, then $ua = vb$, $a^2 = au^*ua = bv^*vb = b^2$, so $a = b$ by uniqueness of positive square roots, and thus $u = v$. The maps $x \mapsto u$ and $x \mapsto a$ are given by continuous formulas.

I.1.6.47. Since the set of invertible positive elements is convex, the a can be linearly moved to 1. There is thus a deformation retraction from A^{-1} to $\mathcal{U}(A)$.

I.1.7. Spatial Tensor Products

Although tensor products of C*-algebras will be discussed in more detail later (), we record here the construction of the minimal or spatial tensor product, which is another important operation used to manufacture C*-algebras.

Tensor Products of Hilbert Spaces

I.1.7.1. Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. We can form the algebraic tensor product $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}'$ (), which is a vector space over $\mathbb{C}$ we denote $\mathcal{H} \odot \mathcal{H}'$ ($\odot$ is standard notation in Operator Algebras for algebraic tensor products over $\mathbb{C}$). There is a well-defined inner product on $\mathcal{H} \odot \mathcal{H}'$ defined on elementary tensors by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}} \langle \eta_1, \eta_2 \rangle_{\mathcal{H}'}$$

and we define the *Hilbert space tensor product* $\mathcal{H} \otimes \mathcal{H}'$ to be the completion.

If $\{\xi_j\}$ and $\{\eta_k\}$ are orthonormal bases for $\mathcal{H}$ and $\mathcal{H}'$ respectively, then $\{\xi_j \otimes \eta_k\}$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{H}'$. In particular, if $\mathcal{H}$ and $\mathcal{H}'$ are finite-dimensional, of dimensions m and n, then $\mathcal{H} \otimes \mathcal{H}'$ is finite-dimensional, of dimension mn, and if $\mathcal{H}$ and $\mathcal{H}'$ are separable, so is $\mathcal{H} \otimes \mathcal{H}'$.

I.1.7.2. If $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H}')$, there is a well-defined bounded operator $S \otimes T$ on $\mathcal{H} \odot \mathcal{H}'$ defined by

$$(S \otimes T)\left(\sum_{k=1}^{n} \xi_k \otimes \eta_k\right) = \sum_{k=1}^{n} S\xi_k \otimes T\eta_k$$

which therefore extends to a bounded operator on $\mathcal{H} \otimes \mathcal{H}'$, also denoted $S \otimes T$; we have $\|S \otimes T\| = \|S\|\|T\|$. We have $(S_1 \otimes T_1)(S_2 \otimes T_2) = S_1S_2 \otimes T_1T_2$ and $(S \otimes T)^* = S^* \otimes T^*$. This extends to a well-defined injective *-homomorphism from the *-algebra $\mathcal{B}(\mathcal{H}) \odot \mathcal{B}(\mathcal{H}') := \mathcal{B}(\mathcal{H}) \otimes_{\mathbb{C}} \mathcal{B}(\mathcal{H}')$ to $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$.

I.1.7.3. There is no canonical embedding of $\mathcal{H}$ (or $\mathcal{H}'$) into $\mathcal{H} \otimes \mathcal{H}'$. However, there are canonical embeddings of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}')$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$, defined by $S \mapsto S \otimes 1$ and $T \mapsto 1 \otimes T$ respectively. These are isometric *-homomorphisms.

I.1.7.4. Definition. If A and B are concrete C*-algebras of operators on $\mathcal{H}$ and $\mathcal{H}'$, then the *spatial tensor product* of A and B is the concrete C*-algebra of operators on $\mathcal{H} \otimes \mathcal{H}'$ which is the norm-closure of

$$A \odot B = \left\{ \sum_{k=1}^{n} S_k \otimes T_k : S_k \in A, T_k \in B \right\} \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H}') \ .$$

The spatial tensor product is denoted $A \otimes B$.

I.1.7.5. If A and B are abstract C*-algebras, then A and B can be represented as concrete C*-algebras of operators, so the spatial tensor product can be constructed. It is a nontrivial fact that the spatial tensor product norm does not depend on how A and B are (faithfully) represented as concrete C*-algebras, so the (spatial) *tensor product* (*C*-tensor product*) $A \otimes B$ can be invariantly defined independent of representation (it can be constructed in a space-free manner too). The notation $A \otimes_{\min} B$ is also used, and the name *minimal tensor product*, for reasons discussed in ().

I.1.7.6. Examples.

(i) If $\mathcal{H}' = \mathbb{C}^n$, then $\mathcal{H} \otimes \mathcal{H}' \cong \mathcal{H}^n$, and

$$\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}') = \mathcal{B}(\mathcal{H}) \odot \mathcal{B}(\mathcal{H}') \cong \mathcal{B}(\mathcal{H}^n) \cong M_n(\mathcal{B}(\mathcal{H}))$$

so if A is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$, then $A \otimes \mathcal{B}(\mathcal{H}') \cong M_n(A)$. Thus, if A is any C*-algebra, $A \otimes \mathbb{M}_n \cong M_n(A)$. In particular, $\mathbb{M}_m \otimes \mathbb{M}_n \cong \mathbb{M}_{mn}$.

(ii) If X is a locally compact Hausdorff space, and B is any C*-algebra, then $C_0(X) \otimes B \cong C_0(X, B)$ (I.1.2.3.). If Y is also locally compact, then $C_0(X) \otimes C_0(Y) \cong C_0(X \times Y)$.

(iii) If $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces, we have $\mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}') = \mathcal{K}(H \otimes \mathcal{H}')$. In particular, we have $\mathbb{K} \otimes \mathbb{M}_n \cong \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$ for any n. [However, if $\mathcal{H}$ and $\mathcal{H}'$ are infinite-dimensional, then $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}')$ is not all of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$.]

I.1.7.7. The tensor product is associative and commutative (up to isomorphism). $A \otimes B$ is unital if and only if both A and B are unital. If B is unital, then $a \mapsto a \otimes 1_B$ is an isometric *-homomorphism, so we can regard A as a C*-subalgebra of $A \otimes B$. Similarly, if A is unital, $b \mapsto 1_A \otimes b$ gives an embedding of B into $A \otimes B$. In general, A and B only sit in the multiplier algebra of $A \otimes B$. The images commute.

I.1.7.8. It is complicated, and not completely known, how to find $\mathrm{Prim}(A \otimes B)$, but if one of A and B is reasonably nice ("exact") we have $\mathrm{Prim}(A \otimes B) \cong \mathrm{Prim}(A) \times \mathrm{Prim}(B)$. But a special case is true in general: if A and B are simple, then $A \otimes B$ is simple.

Stable C*-Algebras and Stable Isomorphism

The following discussion is limited to *separable* C*-algebras (the situation is far more complicated and not nearly as clean for nonseparable C*-algebras).

I.1.7.9. Definition. A separable C*-algebra A is *stable* if $A \cong A \otimes \mathbb{K}$. If A is a separable C*-algebra, the *stable algebra* of A is $A \otimes \mathbb{K}$. Two separable C*-algebras A and B are *stably isomorphic* if $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$, i.e. if the stable algebras of A and B are isomorphic.

Since $\mathbb{K} \cong \mathbb{K} \otimes \mathbb{K}$, the stable algebra of any (separable) C*-algebra is stable, hence the terminology is consistent. Any A is stably isomorphic to its stable algebra. A stable C*-algebra is necessarily nonunital.

I.1.7.10. Example. The stable algebra of any $\mathbb{M}_n$ is $\mathbb{K}$, as is the stable algebra of $\mathbb{K}$. Thus all the $\mathbb{M}_n$ are stably isomorphic to $\mathbb{K}$ and to each other. More generally, if A is any (separable) C*-algebra, all matrix algebras over A are stably isomorphic to each other and to $A \otimes \mathbb{K}$.

I.1.7.11. Stably isomorphic C*-algebras are the "same up to size"; stable C*-algebras are the C*-algebras of "uniformly infinite size." There is another notion called *(strong) Morita equivalence*, which is defined as a functorial equivalence of representation theories obtained by tensoring with a certain bimodule. Two separable (or, more generally, σ-unital) C*-algebras are Morita equivalent if and only if they are stably isomorphic (Brown-Green-Rieffel Theorem; cf. [Bla06, II.7.6.11]).

I.1.7.12. Here is a way of visualizing the stable algebra of a C*-algebra. Fix a C*-algebra A. Form the inductive system

$$A \to M_2(A) \to M_3(A) \to \cdots \to M_n(A) \to M_{n+1}(A) \to \cdots$$

where $M_n(A)$ is embedded in $M_{n+1}(A)$ as $a \mapsto diag(a, 0)$, i.e. expanding a matrix by adding a row and column of zeroes. Note that this is a nonunital embedding. The algebraic direct limit, denoted $M_\infty(A)$, can be regarded as the set of infinite matrices with entries in A and only finitely many nonzero entries. $M_\infty(A)$ has a natural structure as a (nonunital) *-algebra, and a norm, and $A \otimes \mathbb{K}$ is the completion. $A \otimes \mathbb{K}$ can also be regarded as an algebra of infinite matrices over A, although it is difficult if not impossible to explicitly describe which matrices are in this algebra.

Corners

I.1.7.13. If A is a C*-algebra and p is a (nonzero) projection in A, then the set pAp is a C*-subalgebra of A, called the *p-corner* of A. It is a *full corner* if p is not contained in any nontrivial closed ideal of A (if A is simple, every corner in A is full). The C*-algebra pAp has a unit p whether or not A is unital; but note that even if A is unital, the statement "pAp is a unital C*-subalgebra of A" is ambiguous since the unit of pAp is not the same as the unit of A. Any corner in a simple C*-algebra is simple.

I.1.7.14. A full corner in a (separable) C*-algebra A is stably isomorphic to A. This is not directly obvious, but it is essentially immediate that they are Morita equivalent. In fact, two unital C*-algebras are stably isomorphic if and only if one (equivalently, each) is isomorphic to a full corner in a matrix algebra over the other ().

Infinite Tensor Products

I.1.7.15. Let (A_n) be a sequence of *unital* C*-algebras. Set $B_n = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ for each n. Embed B_n into $B_{n+1} \cong B_n \otimes A_{n+1}$ by $b \mapsto b \otimes 1_{A_{n+1}}$. The inductive limit of the B_n with these (unital) embeddings is called the *infinite tensor product* of the A_n. If all the A_n's are the same A, the infinite tensor product is the *infinite tensor power* of A, denoted A^∞ or $A^{\otimes\infty}$. If all the A_n are separable [resp. simple], then the infinite tensor product is separable [resp. simple].

I.1.7.16. Example. Let $A = \mathbb{M}_2$. Then $A^{\otimes\infty}$ is an infinite-dimensional simple unital C*-algebra called the *CAR algebra* or *Fermion algebra*. CAR stands for the "Canonical Anticommutation Relations" of quantum mechanics; the CAR algebra arises naturally in representations of these relations (see e.g. [Tak03b, Exercise XIV.1] for a discussion). A is a UHF algebra (I.3.1.1.).

I.1.7.17. Uncountable tensor products can be defined similarly, but they are never separable except in degenerate cases. An infinite tensor product can be defined if all but finitely many of the algebras are unital. An infinite tensor product of nonunital C*-algebras cannot be defined in general, but infinite tensor products can be defined in somewhat greater generality [Bla77a]: if $\{A_i : i \in I\}$ is an indexed collection of C*-algebras, and p_i is a (nonzero) projection in A_i for each i (or for all but finitely many i), then $\bigotimes_{i\in I}(A_i, p_i)$ can be formed by embedding finite tensor products in larger finite tensor products by tensoring with the projections in the additional factors. The usual infinite tensor product is the case where $p_i = 1$ for all i. The resulting C*-algebra depends on the choice of the p_i, and is nonunital if infinitely many p_i are not 1 (or if even one of the A_i is nonunital).

I.2. Von Neumann Algebras and Factors

Von Neumann algebras, originally called *Rings of Operators*, were the first C*-algebras to be systematically studied, beginning with von Neumann in the late 1920s (indeed, in his first paper on the subject [vN30a], he gave the definition of an abstract Hilbert space for the first time). Von Neumann algebras are rather more tractable than general C*-algebras and the theory is now quite advanced, although far from complete, and the classification of injective factors with separable predual can serve as an illuminating (but to some extent misleading) model for the classification of separable simple nuclear C*-algebras.

I.2.1. General Theory

I.2.1.1. Definition. A *von Neumann algebra* is a unital *-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ which is closed in the weak operator topology.

I.2.1.2. Recall that $\mathcal{B}(\mathcal{H})$ has several natural topologies which are weaker than the norm topology (strictly weaker and all distinct if $\mathcal{H}$ is infinite-dimensional): the strong operator topology, the weak operator topology, the σ-strong operator topology, the σ-weak operator topology, the strong-* operator topology, the σ-strong-* operator topology, the Banach space weak topology (rarely used), and perhaps others too. There is another natural topology: $\mathcal{B}(\mathcal{H})$ is naturally the dual space of the Banach space $\mathcal{L}^1(\mathcal{H})$ of trace-class operators, and as such has a corresponding weak-* topology; but this turns out to be the same as the σ-weak operator topology. Although the six operator topologies are distinct if $\mathcal{H}$ is infinite-dimensional, it turns out that for a *-algebra of operators closure in all the operator topologies is the same thing.

Von Neumann algebras are in particular norm-closed and are thus concrete C*-algebras; but closure in the weak topology forces them to have very special properties (to begin with, they are always unital).

I.2.1.3. There is an algebraic characterization of von Neumann algebras. If S is a subset of $\mathcal{B}(\mathcal{H})$, write S' for the set of operators in $\mathcal{B}(\mathcal{H})$ which commute with every element of S. If S is closed under the adjoint, then S' is a von Neumann algebra since addition and multiplication are separately weakly continuous. S' is called the *commutant* of S. We can then take the commutant $S'' = (S')'$, which is a von Neumann algebra containing S, called the *bicommutant* of S. (The hierarchy stops here: we have $S''' = S'$.)

Von Neumann's *Bicommutant Theorem* [vN30b] was the first (1928) real theorem in the subject of Operator Algebras:

I.2.1.4. Theorem. [Bicommutant Theorem] Let M be a unital *-subalgebra of $\mathcal{B}(\mathcal{H})$. The following are equivalent:

(i) M is σ-strong-* closed.

(ii) M is weakly closed, i.e. a von Neumann algebra.

(iii) $M = M''$.

In particular, if S is a subset of $\mathcal{B}(\mathcal{H})$ which is closed under the adjoint, then the smallest von Neumann algebra on $\mathcal{H}$ containing S is S''.

We could have listed more equivalent conditions specifying that M is closed under topologies between the weak and the σ-strong-*, e.g. the strong and the σ-weak.

I.2.1.5. The requirement that a von Neumann algebra on $\mathcal{H}$ contains the unit of $\mathcal{B}(\mathcal{H})$ is technically necessary to make the Bicommutant Theorem work (since $1 \in S'$ for every set S); but it is essentially superfluous. If M is a weakly closed *-subalgebra of $\mathcal{B}(\mathcal{H})$, then the set $\{a\xi : a \in M, \xi \in \mathcal{H}\}$ is a closed subspace $\mathcal{X}$ of $\mathcal{H}$, the *essential subspace* of M, invariant under M, and M acts as a von Neumann algebra on $\mathcal{X}$ (and zero on $\mathcal{X}^\perp$). In particular, as a C*-algebra M has a unit, the projection onto $\mathcal{X}$.

I.2.1.6. Von Neumann algebras have a *lot* of projections. In fact, if x is a normal element of a von Neumann algebra M (e.g. self-adjoint or unitary), then all the spectral projections of x are also in M; and the projections in M form a complete lattice. A von Neumann algebra not only has continuous functional calculus, but even *measurable functional calculus.* If C*-algebras can be thought of as "noncommutative topological spaces," then von Neumann algebras can be (and are) thought of as "noncommutative measure spaces." Indeed, a commutative von Neumann algebra is of the form $L^\infty(X, \mu)$ for some measure space (X, μ) (which is far from unique).

I.2.1.7. Since $\mathcal{B}(\mathcal{H})$ is the dual of a Banach space ($\mathcal{L}^1(\mathcal{H})$), and the weak-* topology is the σ-weak operator topology, any subspace of $\mathcal{B}(\mathcal{H})$ which is closed in the σ-weak operator topology, in particular any von Neumann algebra on $\mathcal{H}$, is also the dual of a Banach space (a quotient of $\mathcal{L}^1(\mathcal{H})$). It turns out that this is the *unique* (up to isometric isomorphism) such Banach space, and it is called the *predual* of M, denoted M_*; the weak-* topology on M from M_* (the σ-weak topology) is thus intrinsic to the Banach space structure of M. M_* can be described as the subspace of M^* consisting of the "normal" (σ-weakly continuous) linear functionals on M.

Being the dual of a Banach space actually characterizes von Neumann algebras among C*-algebras: a C*-algebra A is (isometrically) isomorphic to a von Neumann algebra if and only if it is the dual of some Banach space. Such C*-algebras are called *W*-algebras.* Some authors use the names "von Neumann algebra" and "W*-algebra" synonymously, but for most of us there is a technical difference: a W*-algebra is an abstract C*-algebra which can be represented as a von Neumann algebra. The distinction is relatively unimportant, though, since, unlike in the C*-algebra case, the representation of a W*-algebra as a von Neumann algebra is essentially unique ("unique up to multiplicity").

I.2.1.8. Any finite-dimensional concrete C*-algebra (containing the unit) is a von Neumann algebra, and hence any finite-dimensional C*-algebra is a W*-algebra (in fact, they are exactly the C*-algebras which are reflexive as Banach spaces).

An infinite-dimensional von Neumann algebra can never be norm-separable. But the following are equivalent for a von Neumann algebra (or W*-algebra) M:

(i) M_* is separable (M has *separable predual*).

(ii) M is separable in the weak operator topology.

(iii) M is separable in the σ-strong-* operator topology.

(iv) The closed unit ball of M is metrizable in the weak operator topology.

(v) M is isomorphic to a von Neumann algebra on a separable Hilbert space.

Thus von Neumann algebras with separable predual are the appropriate von Neumann algebra analogs of separable C*-algebras. In some references such as [Tak02]–[Tak03b], such von Neumann algebras are called "separable," but this term is too ambiguous for us so we will generically use "separable predual."

I.2.1.9. The weakly closed ideals in a von Neumann algebra are easy to describe: they are just direct summands (i.e. principal ideals generated by central projections). The norm-closed ideals, though, can be quite complicated.

Factors

I.2.1.10. DEFINITION. A *factor* is a von Neumann algebra whose center is $\mathbb{C}1$.

Factors are the "simple von Neumann algebras," the ones with no nontrivial weakly closed ideals.

I.2.1.11. General von Neumann algebras, at least in the separable predual case, are built up from factors in a more direct way than general C*-algebras are assembled from simple ones: every von Neumann algebra is a "measurable direct sum" (*direct integral*) of factors (the center of a von Neumann algebra is a commutative von Neumann algebra, hence of the form L^∞ of a measure space; the direct integral is over this measure space). The direct integral was developed in [vN49].

Thus understanding general von Neumann algebras (at least ones with separable predual) reduces in a fairly straightforward way to understanding factors.

I.2.2. Type Theory

MURRAY and VON NEUMANN initiated the study and classification of factors in a series of papers in the 1930s and 1940s. The first main step was to divide factors into three classes.

I.2.2.1. There is one obvious class of factors: $\mathcal{B}(\mathcal{H})$ itself. These are called *Type I* factors. They depend up to isomorphism on the dimension of $\mathcal{H}$, and are finite if $\mathcal{H}$ is finite-dimensional, properly infinite if $\mathcal{H}$ is infinite-dimensional. A finite Type I factor is isomorphic to $\mathbb{M}_n$ for some n; $\mathbb{M}_n$ is called a *Type* I_n *factor*. $\mathcal{B}(\mathcal{H})$ is called a *Type* I_∞ *factor* if $\mathcal{H}$ is infinite-dimensional. So there is a unique type I_n factor for each $n \in \mathbb{N}$, and a unique Type I_∞ factor with separable predual. These are precisely the factors which contain a minimal nonzero projection.

I.2.2.2. It is not at all obvious there are factors which are not Type I. MURRAY and VON NEUMANN said that

> A factor is *Type II* if it contains a nonzero finite projection but no minimal projections.
>
> A factor is *Type III* if all nonzero projections are infinite (hence properly infinite).

They furthermore divided the Type II factors into two classes: *Type* II_1 if the factor is finite, i.e. contains no infinite projections, and *Type* II_∞ otherwise.

I.2.2.3. The same definition of (pure) type can be given for von Neumann algebras which are not factors, with "minimal projection" replaced by "abelian projection"; a von Neumann algebra is of Type j ($j = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$) if and only if all the factors in its central direct integral decomposition are of type j. A general von Neumann algebra M has unique mutually orthogonal central projections p_j adding to 1 (some may be 0) with p_jM of pure type j. A Type I von Neumann algebra can be decomposed uniquely as a C*-direct product of von Neumann algebras of pure type I_n ($1 \leq n \leq \infty$). A von Neumann algebra is commutative if and only if it is pure Type I_1. Type I and Type II von Neumann algebras are "semifinite" and Type III von Neumann algebras are "purely infinite."

I.2.2.4. Murray and von Neumann showed in their first paper [MVN36] in 1935 that factors of types II_1 and II_∞ exist on a separable infinite-dimensional Hilbert space. They described non-type-I factors as "pathological"; according to http://jeff560.tripod.com/mathword.html, this was the first known use of the term *pathological* in a mathematics publication. Type III factors were shown to exist in [vN40].

The first non-type-I factors they constructed were via the *group measure space construction*, a von Neumann algebra version of transformation group C*-algebras. We describe the simplest case, which can be generalized:

I.2.2.5. Example. Let (X, μ) be a finite measure space, and $T : X \to X$ an invertible measure-preserving transformation. Then T defines a unitary operator U_T on $L^2(X, \mu)$ by $U_T(f) = f \circ T$. Let M be the von Neumann algebra generated by U_T and $L^\infty(X, \mu)$ (acting as multiplication operators). If T is ergodic (Y a measurable subset of X, $T(Y) = Y$ implies $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$) but not transitive (i.e. the orbit of every point has measure zero), then M is a Type II_1 factor. There are many such examples of transformations, e.g. a rotation of a circle by an irrational multiple of 2π (cf. ()).

I.2.2.6. A related construction, which first appeared in [vN40], which is a generalization from the right point of view, has been consistently important to the present. Let G be a countable group, and (X, μ) a measure space. Let $g \mapsto T_g$ be a homomorphism of G into the group of invertible measurable (not necessarily measure-preserving) transformations of X. Then $\phi \mapsto \phi \circ T_g$ defines an automorphism α_g of $L^\infty(X, \mu)$. Form the Hilbert space $\mathcal{H} = \ell^2(G, L^2(X, \mu))$, the Hilbert space of square-summable sequences (indexed by G) of vectors in $L^2(X, \mu)$. Let $\phi \in L^\infty(X, \mu)$ act on the g'th summand by multiplication by $\alpha_g(\phi)$, and let G act on the summands by left translation of the index (this is an action λ of G by unitary operators). Let M be the von Neumann algebra generated by $L^\infty(X, \mu)$ and $\lambda(G)$. If the action of G is free () and ergodic, but not transitive, then M is a non-Type-I factor, which has separable predual if G is countable and (X, μ) is reasonable.

M is Type II_1 if there is a finite invariant measure equivalent to μ (this is essentially the previous case if $G = \mathbb{Z}$).

M is Type II_∞ if there is a semifinite invariant measure equivalent to μ, but no finite invariant measure.

M is Type III if there is no semifinite invariant measure equivalent to μ.

Examples of all types can be exhibited.

I.2.2.7. As a variation, consider the case where X is a single point but G is a complicated group. Then G acts on $\ell^2(G)$ by left translation as unitary operators; this is called the *left regular representation* of G. Let $L(G)$ be the von Neumann algebra generated by the unitary operators of left translation; this is called the (left) von Neumann algebra of G. It has a faithful trace, hence is finite. It is not difficult to show that if all the conjugacy classes in G other than the identity are infinite (such a group is called an *ICC group*), then $L(G)$ is a factor, which is necessarily Type II_1. There are many examples of ICC groups, such as:

Free groups $\mathbb{F}_n$ on more than one generator ($2 \leq n \leq \infty$).

The group $S_f(\mathbb{N})$ of finite permutations of $\mathbb{N}$ (permutations moving only finitely many elements).

Murray and von Neumann were able to show [MvN43], using a central sequence argument, that the factor(s) $L(\mathbb{F}_n)$ are not isomorphic to the factor $L(S_f(\mathbb{N}))$, and thus there are at least two nonisomorphic Type II_1 factors on a separable Hilbert space. (It is an open problem to this day whether the $L(\mathbb{F}_n)$ for different n are isomorphic.) In 1969, D. McDuff [McD69] exhibited (using more sophisticated central sequence arguments) an uncountable family of countable ICC groups for which the group von Neumann algebras are pairwise nonisomorphic, so there are uncountably many ($2^{\aleph_0}$) distinct Type II_1 factors on a separable Hilbert space. (Before this result various people had labored mightily to produce a small [< 10] finite number of nonisomorphic II_1 factors with separable predual by *ad hoc* constructions, and similarly for Type III factors before [Pow67].) The construction can be done also for nondiscrete locally compact G, and there exist second countable groups G for which $L(G)$ is a Type III factor [Bla77b] (cf. [BC95]).

I.2.2.8. A factor with a trace (tracial state) must be finite, hence Type II_1 if infinite-dimensional. In [MVN36], Murray and von Neumann attempted to show that every Type II_1 factor has a trace, but were not quite able to do it: they showed that a Type II_1 factor M has a unique quasitrace () τ, and that it parametrizes equivalence and subordination of projections: if p and q are projections in M, then $p \sim q$ if and only if $\tau(p) = \tau(q)$, and $p \precsim q$ if and only if $\tau(p) \leq \tau(q)$, and furthermore that τ takes all values in $[0, 1]$ on projections. In their second paper [MvN37], they showed by a spatial argument that the quasitrace τ is indeed always a trace.

I.2.2.9. Thus a Type II_1 factor can be thought of as looking like a matrix algebra, but instead of having a finite discrete set of rows and columns it has a "continuous" (actually, more accurately, "measurable") set of rows and columns, one for each number in $[0, 1]$. Von Neumann thought of these as defining a "continuous geometry" where dimensions of subspaces were not discrete but varied continuously through $[0, 1]$, and wrote a book on the subject [vN98]. Similar comments apply to Type II_∞ and Type III factors, which are also called "continuous factors," although they look less like finite matrix algebras.

I.2.2.10. Although factors are "simple" as von Neumann algebras, they are not necessarily simple as C*-algebras. Type I_n ($n \in \mathbb{N}$) and Type II_1 factors are simple C*-algebras, as are countably decomposable Type III factors (every collection of pairwise orthogonal nonzero projections is countable). Type I_∞ and Type II_∞ factors are not simple as C*-algebras (the finite projections generate a nontrivial ideal), and non-countably-decomposable Type III factors (which can only occur on nonseparable Hilbert spaces) are also not simple as C*-algebras. Thus, among factors with separable predual, exactly the Type II_1 and Type III factors, as well as the finite-dimensional factors, are simple as C*-algebras.

Tensor Products of von Neumann Algebras

I.2.2.11. Let M and N be von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively. The *von Neumann tensor product* $M \bar{\otimes} N$ is the von Neumann algebra on $\mathcal{H} \otimes \mathcal{H}'$ generated by $M \odot N$, i.e. $(M \odot N)''$. $M \bar{\otimes} N$ contains the C*-tensor product $M \otimes N$, but they are not equal unless M or N is finite-dimensional (in which case they are both just the algebraic tensor product). We have $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{B}(\mathcal{H}') = \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$.

I.2.2.12. Because of the essential uniqueness of the representation of a von Neumann algebra, $M \bar{\otimes} N$ does not depend up to isomorphism on the representations of M and N as von Neumann algebras; it can even be defined in a space-free manner, and thus could be called the *W*-tensor product*.

I.2.2.13. The center $\mathcal{Z}(M \bar{\otimes} N)$ of $M \bar{\otimes} N$ is $\mathcal{Z}(M) \bar{\otimes} \mathcal{Z}(N)$. In particular, $M \bar{\otimes} N$ is a factor if (and only if) M and N are factors.

I.2.2.14. The following type rules apply to W*-tensor products:

(finite) $\bar{\otimes}$ (finite) = finite.

(semifinite) $\bar{\otimes}$ (semifinite)= semifinite.

(properly infinite) $\bar{\otimes}$ (anything) = properly infinite.

(Type I) $\bar{\otimes}$ (Type I) = Type I.

(semifinite) $\bar{\otimes}$ (Type II) = Type II.

(anything) $\bar{\otimes}$ (Type III) = Type III.

In particular,

(Type II_1) $\bar{\otimes}$ (Type II_1) = Type II_1.

(Type II_1) $\bar{\otimes}$ (Type I_∞) = Type II_∞.

I.2.2.15. If M is a von Neumann algebra on $\mathcal{H}$ and p is a nonzero projection in M, then the corner pMp is a von Neumann algebra on $p\mathcal{H}$. If M is a factor, then pMp is also a factor, of the same type (I, II, III) as M; if M is semifinite and p is a finite projection, then pMp is finite. Thus if M is II_∞ and $p \in M$ is finite (and nonzero), then pMp is II_1. If M is a properly infinite factor and p is a (nonzero) projection in M, then $M \cong pMp \bar{\otimes} \mathcal{B}(\mathcal{H}')$, where the dimension of $\mathcal{H}'$ is the maximum number of pairwise orthogonal projections in M equivalent to p. In particular, if M is a Type II_∞ factor with separable predual and p is a finite nonzero projection in M, then $N = pMp$ is Type II_1 and $M \cong N \bar{\otimes} \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is separable and infinite-dimensional. Thus every II_∞ factor is the W*-tensor product of a II_1 factor and a I_∞ factor. The N is uniquely determined by M up to stable isomorphism, but not isomorphism in general since if N is Type II_1 and q is a (nonzero) projection in N, then qNq is not necessarily isomorphic to N.

I.2.2.16. Although finite W*-tensor products of von Neumann algebras are well defined, the same is not true for infinite tensor products since there is no well-defined notion of inductive limit for W*-algebras, in contrast with the C*-tensor product. (This is related to the fact that infinite tensor products of Hilbert spaces cannot be defined without choosing some distinguished vectors.)

McDuff Factors

A tensor product property which was important in McDuff's work on II_1 factors and in subsequent factor theory, and which has a crucial analog in the C*-case, is the following:

I.2.2.17. Definition. A factor M is a *McDuff factor* if $M \cong M \bar{\otimes} \mathcal{R}$ (I.2.3.23.).

A McDuff factor cannot be Type I; the term is used primarily for II_1 factors.

I.2.3. Approximately Finite Dimensional Factors

In this section, we describe the classification of approximately finite dimensional factors with separable predual, and several other natural and important characterizations of this class.

I.2.3.1. Definition. Let M be a von Neumann algebra with separable predual. M is *approximately finite dimensional*, or *AFD*, if every finite subset of M can be arbitrarily well approximated in the weak operator topology by elements in a finite-dimensional C*-subalgebra of M.

I.2.3.2. We have deliberately restricted the definition of AFD to von Neumann algebras with separable predual, although the definition makes perfect sense in general. We did this for economy of language, to avoid having to repeatedly write "separable predual."

I.2.3.3. The term *hyperfinite* is often used as a synonym for AFD. This is fine for finite von Neumann algebras, e.g. II_1 factors, but I find it objectionable in general since it is language abuse to have an algebra which is hyperfinite but not finite (even though the "finite" is used in two different senses). We will use "hyperfinite" only for II_1 factors.

The definition of AFD is equivalent to the following apparently more restrictive condition:

I.2.3.4. Proposition. A von Neumann algebra M is AFD if and only if there is an increasing sequence of finite-dimensional unital C*-subalgebras of M whose union is weakly dense in M.

In both the definition and this proposition, density in the weak operator topology can be replaced by density in the σ-strong-* topology (or in any of the intermediate operator topologies).

I.2.3.5. It is easily seen (e.g. by representations of the CAR algebra) that any Type I von Neumann algebra with separable predual is AFD, and that a von Neumann tensor product of two AFD von Neumann algebras is AFD.

I.2.3.6. AFD von Neumann algebras are the smallest natural class to consider beyond the Type I von Neumann algebras. Type I C*-algebras are precisely the C*-algebras whose representations generate only Type I von Neumann algebras (although note that most Type I von Neumann algebras are not Type I C*-algebras). There is a large class of C*-algebras (the nuclear C*-algebras ()) whose representations generate nothing but AFD von Neumann algebras. Conversely, we have the following result of O. Maréchal [Mar75], which shows that the AFD von Neumann algebras are the next class:

I.2.3.7. Theorem. Let A be a separable C*-algebra which is not Type I, and let M be a properly infinite AFD von Neumann algebra on a (separable) Hilbert space $\mathcal{H}$. Then there is a representation π of A on $\mathcal{H}$ such that $\pi(A)'' = M$. (In other words, M has a weakly dense C*-subalgebra isomorphic to a quotient of A.)

In particular, every properly infinite AFD von Neumann algebra contains a weakly dense unital C*-subalgebra isomorphic to the CAR algebra. Describing the finite von Neumann algebras generated by representations of a C*-algebra A is more delicate, and connected with the trace space $\mathcal{T}(A)$.

Injective von Neumann Algebras

I.2.3.8. Definition. A von Neumann algebra M is *injective* if, for every C*-algebra A and C*-subalgebra B of A, every completely positive contraction (cpc) from B to M extends to a cpc from A to M (). In other words, M is an injective C*-algebra, an injective object in the category of C*-algebras and completely positive contractions:

$$\begin{tikzcd} A \arrow[dr, dotted] & \\ B \arrow[u, hook] \arrow[r] & M \end{tikzcd}$$

I.2.3.9. By adding units, to show that M is injective it suffices to show that if A is a unital C*-algebra, B is a unital C*-subalgebra of A, every unital completely positive (ucp) map from B to M extends to a ucp map from A to M.

I.2.3.10. It is a somewhat nontrivial theorem (a special case of the Arveson Extension Theorem) that any finite-dimensional C*-algebra is injective, and thus using the weak compactness of the unit ball that an AFD von Neumann algebra is injective. A von Neumann algebra is injective if and only if all factors in its central direct integral decomposition are injective.

I.2.3.11. If $M \subseteq \mathcal{B}(\mathcal{H})$ is injective, then taking $A = \mathcal{B}(\mathcal{H})$ and $B = M$, the identity map on M extends to an idempotent ucp map (a *conditional expectation*) from $\mathcal{B}(\mathcal{H})$ onto M. Conversely, if there is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto a C*-subalgebra M, then since a Type I factor is injective it follows that M is also injective. So a von Neumann algebra (or concrete C*-algebra) M on $\mathcal{H}$ is injective if and only if there is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto M.

I.2.3.12. An injective C*-algebra is not necessarily a W*-algebra, although it is "almost" a W*-algebra (an *AW*-algebra*). A simple AW*-algebra (in particular, an injective simple C*-algebra) which can be represented on a separable Hilbert space is a W*-algebra (factor). See ().

Approximation by Finite-Rank UCP Maps

I.2.3.13. A von Neumann algebra M is said to have *the weak completely positive approximation property (WCPAP)* if there is a net $\{(\alpha_\lambda, \beta_\lambda) : \lambda \in \Lambda\}$ of normal ucp maps $\alpha_\lambda : M \to \mathbb{M}_{n_\lambda}$ and $\beta_\lambda : \mathbb{M}_{n_\lambda} \to M$ such that $(\beta_\lambda \circ \alpha_\lambda)(x) \to x$ in the weak operator topology for every $x \in M$, i.e. $\beta_\lambda \circ \alpha_\lambda$ converges to the identity map ι_M in the point-weak topology. In other words, we say the diagrams

$$\begin{tikzcd} M \arrow[rr, "\iota_M"] \arrow[dr, "\alpha_\lambda"'] & & M \\ & \mathbb{M}_{n_\lambda} \arrow[ur, "\beta_\lambda"'] & \end{tikzcd}$$

asymptotically commute in the point-weak topology.

I.2.3.14. An injective von Neumann algebra has the WCPAP (with the index set $\Lambda = \mathbb{N}$), although there are unexpectedly subtle and nontrivial technicalities in the proof even in the AFD case (cf. [Bla06, IV.2.4.7]). In fact, the WCPAP is a sort of "order-theoretic approximate finite-dimensionality."

Semidiscrete von Neumann Algebras

I.2.3.15. Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$, with commutant M'. The map

$$\sum_{k=1}^{n} x_k \otimes y_k \mapsto \sum_{k=1}^{n} x_k y_k$$

is a well-defined *-homomorphism from $M \odot M'$ to $\mathcal{B}(\mathcal{H})$, injective if M is a factor. M is said to be *semidiscrete* if this map is continuous for the C*-tensor product norm, i.e. extends to a homomorphism from $M \otimes M'$ to $\mathcal{B}(\mathcal{H})$. (It never extends to $M \bar{\otimes} M'$ unless M is Type I.)

I.2.3.16. A von Neumann algebra is semidiscrete if and only if all factors in its central direct integral decomposition are semidiscrete. The term "semidiscrete" is most commonly used only for factors.

I.2.3.17. If M is a Type I ("discrete") factor on $\mathcal{H}$, then $\mathcal{H}$ splits as a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ in such a way that $M = \mathcal{B}(\mathcal{H}_1) \otimes 1$, so $M' = 1 \otimes \mathcal{B}(\mathcal{H}_2)$ and thus M is semidiscrete. This is the origin of the terminology.

I.2.3.18. Since a finite-dimensional von Neumann algebra is Type I and hence semidiscrete, it appears that an AFD von Neumann algebra is semidiscrete; this turns out to be true, but is subtle to prove. It is relatively easy to show that a von Neumann algebra is semidiscrete if and only if it has the WCPAP (cf. [Bla06, IV.2.4.4]).

I.2.3.19. The definition of semidiscreteness is symmetric in M and M', and thus M is semidiscrete if and only if M' is semidiscrete. There is in fact a deep duality between a von Neumann algebra M and its commutant M'. It is not hard to show that if M is (pure) Type I, II, or III, then M' is also pure of the same type, although subscripts can be different, e.g. if M is Type II$_1$, then M' is Type II, but may be Type II$_1$ or Type II$_\infty$ (or a combination in the non-factor case). In fact, M' is always anti-isomorphic to a full corner of an amplification of M, the von Neumann algebra analog of stable (anti-)isomorphism.

Amenable von Neumann Algebras

There is another equivalent condition, which we merely state here without explanation (cf. ()):

I.2.3.20. Definition. Let M be a von Neumann algebra. M is *amenable (W*-amenable)* if every derivation from M to a dual normal M-bimodule is inner.

See [**?**] for an exposition of what this means, why it is interesting and important, and why it is called "amenability." (An amenable von Neumann algebra is "cohomologically trivial.") Part of the answer is the following theorem ([**?**], [**?**]; cf. [**?**, 12.4.38]):

I.2.3.21. Theorem. Let M be a von Neumann algebra. The following are equivalent:

(i) M is W*-amenable.

(ii) The unitary group $U(M)$ is amenable in the σ-weak (weak, strong) topology.

(iii) There is a discrete amenable group G of unitaries in M with $G'' = M$.

Equivalence of the Conditions

One of the two big theorems of [Con76] is the equivalence of these conditions:

I.2.3.22. Theorem. Let M be a von Neumann algebra on a separable Hilbert space (i.e. with separable predual). The following are equivalent:

(i) M is injective.

(ii) M is semidiscrete.

(iii) M has the WCPAP.

(iv) M is amenable.

(v) M is AFD.

As indicated above, (v) fairly easily implies the other conditions, but most of the rest of the implications are very deep. (Equivalence of (i) and (iv) was shown in [Con78] and [BP78].)

Classification of AFD Factors

The first classification result, fairly easy in retrospect, is due to Murray and von Neumann [MvN43]:

I.2.3.23. Theorem. There exists a unique (up to isomorphism) hyperfinite (AFD) factor of Type II_1, usually denoted $\mathcal{R}$.

It is a far deeper result, an immediate consequence of I.2.3.23. and I.2.3.22., that $\mathcal{R}$ is the unique *injective* II_1 factor (with separable predual).

I.2.3.24. For existence, it is easy to see that $L(S_f(\mathbb{N}))$ is hyperfinite, since $S_f(\mathbb{N})$ is an increasing union of a sequence of finite subgroups. In fact, if G is a countable ICC group, then $L(G)$ is hyperfinite if (and only if) G is amenable. $\mathcal{R}$ can be constructed more simply by the GNS construction from the CAR algebra using the unique trace.

I.2.3.25. Every Type II_1 factor contains a copy of $\mathcal{R}$ as a weakly closed unital C*-subalgebra (subfactor). On the other hand, every infinite-dimensional subfactor of $\mathcal{R}$ is isomorphic to $\mathcal{R}$. Thus $\mathcal{R}$ is the "smallest" Type II_1 factor. Any II_1 factor stably isomorphic to $\mathcal{R}$ is isomorphic to $\mathcal{R}$.

I.2.3.26. There is also an AFD Type II_∞ factor, $\mathcal{R}\bar{\otimes}\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ separable). It was strongly suspected that this is the unique AFD factor of Type II_∞, but this remained a major open problem in von Neumann algebra theory until the proof of I.2.3.22. in [Con76]. The difficulty is that if M is an AFD factor of Type II_∞, then M can be written as a tensor product $N\bar{\otimes}\mathcal{B}(\mathcal{H})$, where N is Type II_1 (and unique up to stable isomorphism); but it is highly unclear that N must be AFD and hence isomorphic to $\mathcal{R}$. However, it is virtually trivial that N is injective (any corner in an injective C*-algebra is injective), hence isomorphic to $\mathcal{R}$ by I.2.3.22. and I.2.3.23..

I.2.3.27. The Type III situation is more complicated. In 1967, R. Powers [Pow67] constructed an uncountable one-parameter family of pairwise nonisomorphic Type III AFD factors, now usually called $\{\mathcal{R}_\lambda : 0 < \lambda < 1\}$, by infinite tensor products (Powers' indexing was different from the modern indexing due to Araki-Woods and Connes, cf. I.2.3.29.).

Powers' results giving an uncountable family of Type III factors were approximately simultaneous with McDuff's construction of uncountably many Type II_1 factors (I.2.2.7.), and the two results were initially regarded as similar and supplementary; but in retrospect they were very different: the Powers factors are AFD, whereas at most one of the McDuff factors can be AFD.

I.2.3.28. The Powers construction was generalized by Araki and Woods [AW69], again using infinite tensor products of matrix algebras with more general product states; they called the factors obtained *ITPFI factors* (standing for "infinite tensor product of finite Type I"; such terminology was famously ridiculed in [Ped79, 6.2.13]). Araki and Woods obtained, in addition to the AFD Type I, Type II_1, and Type II_∞ and the Powers factors, some additional AFD Type III factors: one called $\mathcal{R}_1$, which is just a tensor product of two Powers factors with incommensurable λ, and a family of other factors now called Type III_0. In the process, they introduced the invariants used by Connes in [Con73]. See [Woo82] for a survey.

I.2.3.29. Connes [Con73] codified the invariants used by Powers, Araki, and Woods using the modular automorphism group of a Type III factor (a canonical action of $\mathbb{R}$), and divided Type III factors into Types III_λ, $0 \le \lambda \le 1$. The Powers factor $\mathcal{R}_\lambda$ is Type III_λ, and the Araki-Woods factor $\mathcal{R}_1$ is Type III_1. The additional Type III Araki-Woods factors are indeed of Type III_0 according to Connes' classification.

I.2.3.30. Then, in a technical *tour de force* (he received a Fields medal in 1983 for this work), Connes showed in [Con76] that $\mathcal{R}_\lambda$ is the unique AFD factor of Type III_λ for $0 < \lambda < 1$, and that AFD factors of Type III_0 are classified by ergodic flows (cf. [Kri76]); many, but not all, AFD Type III_0 factors are Araki-Woods factors. One case remained unresolved in [Con76], and was finished by Haagerup in [Haa87]: $\mathcal{R}_1$ is the unique AFD factor of Type III_1. Thus AFD factors were completely classified. Proofs of some of the classification results were improved in [CT77], and alternate arguments given in [Haa89] and [Pop88].

I.2.3.31. The definitive (at least so far) treatment of the classification of AFD factors is [Tak03b], which contains a full discussion and proofs of the results (a fairly complete outline can be found in [Bla06]). We will have little more to say about this classification, which is off the main subject of this book except by (incomplete) analogy. But the connection is really more than an analogy: many of the techniques used to classify separable simple nuclear C*-algebras are analogs and/or adaptations of techniques in the factor case, in good part originating with Connes. In fact, it is the theme of [**?**] that the classifications are both philosophically and technically very similar; I appreciate the point of view he is taking, but I think the best that can be said is probably that the *characterizations* of the classifiable algebras are quite similar and analogous, and many techniques in both the characterization and the proof of the classification theorem are similar, but the actual *classifications* do not bear much resemblance, using completely different invariants. About the only similarity in the classification *per se* is that classifiable separable simple unital nuclear C*-algebras, like factors, split into types: “Type I” (finite-dimensional), “Type II” (infinite-dimensional and stably finite), and “Type III” (purely infinite); the nonunital ones can be split roughly the same way, although the division must be phrased a little differently. And not even all separable simple nuclear C*-algebras fit cleanly into these types, e.g. Rørdam’s example () (which might be called “Type II_∞”, although this is somewhat misleading since it has no trace).

The Non-AFD Case and Further Developments

I.2.3.32. There has been much spectacular recent progress in the structure and classification of non-AFD Type II_1 factors with separable predual, although it’s fair to say there are still more questions than answers about the general classification and no general picture is in sight or even anticipated in the foreseeable future. See [Pop07], [Vae10], [Ioa13] for good surveys. The structure of Type III factors (with separable predual) is more mysterious, although by Connes’ work the study of Type III factors can be reduced to II_∞ factors and then to II_1 factors; a classification of II_1 factors will not directly give a classification of Type III factors or even Type II_∞ factors (the procedure was far from straightforward even in the AFD case!) This work on non-AFD factors has no direct connection with the main subject of this book except possibly as an indication of the direction future work on simple C*-algebras might try to take.

I.2.3.33. Another important and exciting development has been the study of subfactors of a II_1 factor, initiated by Vaughan Jones (he received a Fields medal in 1990 for this work). There are deep and beautiful connections with knot theory and topological quantum field theory. See () for a survey.

Using Factor Classification as a Model

I.2.3.34. One might hope that the classification of separable simple nuclear C*-algebras might follow the pattern of AFD factors. But although there are similarities, the C*-algebra case is *much* more complicated. For example:

(i) The C*-versions of the equivalent conditions of Theorem I.2.3.22. are not equivalent for separable simple C*-algebras, although all but (v), properly phrased, do turn out to be equivalent (the condition is generically called *nuclearity*); however, the various characterizations of (v) are not equivalent in the C*-case. As a related matter, there is a very difficult (and still open) technical problem in the C*-case called the *UCT*, which (as described in [**?**]) has a topological flavor and does not arise in the measure-theoretic context of factors.

(ii) If M is an infinite-dimensional factor with a trace (i.e. a II_1 factor), then the trace is unique and completely determines equivalence and comparability of projections in M, and takes all values in $[0, 1]$ on projections. But a simple unital nuclear C*-algebra can have many traces, and they may not completely determine equivalence or even strict comparability of projections; and a separable C*-algebra can only have countably many equivalence classes of projections, so a trace can take only countably many values in $[0, 1]$ on projections, and this countable set of values can be complicated (in good cases, but not always, it is the intersection of a countable additive subgroup of $\mathbb{R}$ with $[0, 1]$, and any such group containing 1 can occur.).

(iii) A simple unital nuclear C*-algebra need not have "enough" projections to completely determine its structure. In fact, it need not have any nontrivial projections at all!

(iv) A simple nuclear C*-algebra is not necessarily unital, and is not necessarily even stably isomorphic to a unital C*-algebra (i.e. can be stably projectionless).

(v) A purely infinite simple unital nuclear C*-algebra has no analog of the modular automorphism group, and there is no reasonable analog of Type III_λ in the C*-case (although some researchers have been periodically misled into trying to find one), leaving us without the principal classification tool used in the Type III factor case.

Thus there will be no uniqueness in the "Type II_1" C*-case, and fairly complicated invariants will be needed; and any invariants in the "Type III" C*-case will have to be completely different from the factor case. (It should be noted that the classification of separable simple C*-algebras was well underway before the classification of the AFD factors was done, so it is historically inaccurate to say it was modeled on the factor classification, although it has drawn inspiration from the factor classification, and it is pedagogically and conceptually useful to compare and contrast the two classifications.)

A comparison with the commutative case highlights the philosophy that C*-algebras (noncommutative topological spaces) are more varied and complicated than von Neumann algebras (noncommutative measure spaces). There are basically only two infinite-dimensional commutative von Neumann algebras with separable predual, $\ell^\infty = L^\infty(\mathbb{N})$ and $L^\infty(\mathbb{R})$ (the only other ones are the direct sum of these two and direct sums of $L^\infty(\mathbb{R})$ with finitely many copies of $\mathbb{C}$). But there is one separable unital commutative C*-algebra for each compact metrizable space, and it appears impossible to give a reasonable classification of compact metrizable spaces, even compact subsets of the plane, in terms of simpler invariants.

One might conclude that the C*-case is hopeless. But it is not, as we shall see!

I.3. Approximately Homogeneous C*-Algebras

Where to begin with finding and classifying simple C*-algebras? Maybe a good place is to try to find the C*-analog(s) of the hyperfinite II_1 factor. These would at least be separable simple unital C*-algebras with a trace, and hence stably finite (), and they should at least satisfy some C*-version of one or more of the conditions of I.2.3.22. (which one might not expect to be equivalent in the C*-case, and are in fact not).

I.3.1. UHF Algebras

The hyperfinite II_1 factor has an increasing sequence of unital finite-dimensional matrix subalgebras whose union is weakly dense. The most restrictive C*-condition analogous to those of I.2.3.22. is to require such a sequence whose union is norm-dense. Such a C*-algebra is called a *UHF algebra*, where UHF stands for "uniformly hyperfinite." These were the first C*-algebras which were not either Type I or (almost) von Neumann algebras (in particular, the first separable non-type-I C*-algebras) to be systematically studied, by Glimm in 1959 [Gli60]. (But one should not overlook the early work of Calkin [Cal41].)

I.3.1.1. Definition. An infinite-dimensional C*-algebra A is a *UHF algebra* if it is isomorphic to an inductive limit $\lim_{\to}(A_n, \phi_{n,n+1})$, where each A_n is a matrix algebra $\mathbb{M}_{k_n}$ and the connecting maps $\phi_{n,n+1}$ are unital (and necessarily injective since matrix algebras are simple).

The requirement that A be infinite-dimensional merely rules out that A itself is a matrix algebra.

I.3.1.2. A UHF algebra is infinite-dimensional, separable, simple, and unital. UHF algebras are the cleanest direct C*-analogs of the hyperfinite II_1 factor. It turns out that there are uncountably many up to isomorphism (even up to stable isomorphism), but they can be nicely described.

I.3.1.3. Example. We have already seen an example, the CAR algebra (I.1.7.16.). We give here an alternate description of the construction which is arguably more elementary. All other UHF algebras can be constructed in essentially the same way.

Set $A_n = \mathbb{M}_{2^{n-1}}$. Embed A_n into A_{n+1} by

$$\phi_{n,n+1}(a) = diag(a,a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

where the entries in the last matrix are $2^{n-1} \times 2^{n-1}$ blocks. This embedding is easily seen to be an injective unital *-homomorphism, hence an isometry. We let A_∞ be the algebraic direct limit, and A the completion. Then A is the CAR algebra, also called the *UHF algebra of type* 2^∞, denoted $\mathbb{M}_{2^\infty}$.

A_∞ has a standard trace τ since each A_n has a unique trace τ_n and $\tau_{n+1}(\phi_{n,n+1}(a)) = \tau_n(a)$ for all n and a. This τ extends to a trace on A by (uniform) continuity; write τ also for the extended trace. If τ' is a trace on A, then the restriction of τ' to $\phi_n(A_n)$ must be the unique trace on this matrix algebra, i.e. $\tau'|_{A_n} = \tau|_{A_n}$ for each n. Thus $\tau' = \tau$ on the dense subset A_∞ of A, hence $\tau' = \tau$ by continuity. So A has a unique trace τ.

The values τ takes on projections in A_∞ are precisely the values taken in the A_n, namely the dyadic rationals (rational numbers which can be written as a fraction with denominator a power of 2) in $[0,1]$. By II.1.12.5., II.1.8.4., and I.1.6.30., these are also the values taken on projections in A. The trace completely determines equivalence and subordination of projections in A, just as in a II_1 factor.

I.3.1.4. Example. As a variation, let $B_n = \mathbb{M}_{3^{n-1}}$ for each n, and define

$$\phi_{n,n+1}(b) = diag(b,b,b) = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$$

(where the entries are $3^{n-1} \times 3^{n-1}$ blocks) and take B_∞ to be the algebraic direct limit and B the completion. B is called the *UHF algebra of type* 3^∞, denoted $\mathbb{M}_{3^\infty}$. There is again a unique trace τ on B which determines equivalence and subordination of projections. But the values τ takes on projections in B are the triadic rationals (rational numbers which can be written with denominator a power of 3) in $[0,1]$. Thus B cannot be isomorphic to A.

We have an enormous number of variations of the construction, based on the following fact from linear algebra which effectively describes the rather trivial representation theory of a matrix algebra (essentially the same result holds over any field):

I.3.1.5. Proposition. Let $\mathbb{M}_k$ and $\mathbb{M}_n$ be matrix algebras. Then there is a unital homomorphism from $\mathbb{M}_k$ to $\mathbb{M}_n$ if and only if k divides n. In this case, any two unital (*-)homomorphisms are unitarily equivalent. If $n = mk$, then up to unitary equivalence the unique unital homomorphism is given by

$$a \mapsto diag(a,a,\dots,a) = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}$$

where the array is $m \times m$ and each entry is a $k \times k$ block. This is called the *embedding of multiplicity* m. It can be alternately described as the embedding $a \mapsto a \otimes 1$ of $\mathbb{M}_k$ to $\mathbb{M}_k \otimes 1 \subseteq \mathbb{M}_k \otimes \mathbb{M}_m \cong \mathbb{M}_n$.

I.3.1.6. The general construction is done by fixing a strictly increasing sequence (k_n) of natural numbers, each dividing the next, say $k_{n+1} = m_n k_n$ for natural numbers $m_n > 1$, with $k_1 = 1$. Set $A_n = \mathbb{M}_{k_n}$ and $\phi_{n,n+1}$ an embedding of multiplicity m_n for each n. Note that we have $k_{n+1} = \prod_{j=1}^n m_j$ for each n.

I.3.1.7. We can interpret $\boldsymbol{m} = \prod_{j=1}^\infty m_j$ as a *supernatural number*, a formal product

$$2^{e_2} 3^{e_3} 5^{e_5} \cdots$$

where an infinite number of nonzero exponents and infinite exponents are allowed (i.e. there is one $e_p \in \mathbb{Z}_+ \cup \{\infty\}$ for each prime number p). The sum of the exponents is ∞, i.e. $\boldsymbol{m}$ is not a natural number. Conversely, for each supernatural number $\boldsymbol{m}$ there is a sequence (m_j) of natural numbers greater than 1 with $\boldsymbol{m} = \prod_{j=1}^\infty m_j$ (in fact, there are many such sequences).

I.3.1.8. Let A be the UHF algebra constructed as in I.3.1.6., with supernatural number $\boldsymbol{m}$. Then A has a unique trace, and the values the trace takes on projections of A are exactly the set of rational numbers in $[0,1]$ which can be written as fractions whose denominator "divides" $\boldsymbol{m}$ (the values on projections in $\phi_n(A_n)$ are multiples of $\frac{1}{k_n}$ for each n). Thus if B is another UHF algebra with associated generalized integer $\boldsymbol{m}'$, then A and B cannot be isomorphic unless $\boldsymbol{m} = \boldsymbol{m}'$. Since there are uncountably many $(2^{\aleph_0})$ supernatural numbers, there are at least $2^{\aleph_0}$ isomorphism classes of UHF algebras (actually there are exactly $2^{\aleph_0}$ since there are only $2^{\aleph_0}$ isomorphism classes of separable C*-algebras). Thus some number theory (baby number theory, to be sure) comes into the classification of UHF algebras, unlike the case of II_1 factors.

Glimm's main result from [Gli60] is that the associated supernatural number is a *complete* isomorphism invariant for UHF algebras (Glimm did not phrase the result using supernatural numbers; this point of view originated in [Dix67]):

I.3.1.9. Theorem. Let A and B be UHF algebras, with associated supernatural numbers $\boldsymbol{m}$ and $\boldsymbol{m}'$ respectively. Then the following are equivalent:

(i) $A \cong B$.

(ii) $\boldsymbol{m} = \boldsymbol{m}'$.

(iii) For each $n \in \mathbb{N}$, A contains a unital C*-subalgebra isomorphic to $\mathbb{M}_n$ if and only if B does.

As a result, we may write $\mathbb{M}_{\boldsymbol{m}}$ for *the* UHF algebra with supernatural number $\boldsymbol{m}$.

We give the relatively simple proof of this theorem, since the argument is the prototype for many of the later proofs where the technical details are much harder.

Proof: (i) $\Rightarrow$ (iii) is obvious, and (iii) $\Rightarrow$ (ii) by the argument in I.3.1.8.. For (ii) $\Rightarrow$ (i), suppose $\boldsymbol{m} = \boldsymbol{m}'$. Let (m_n) and (m'_n) be the sequences for the construction of A and B as in I.3.1.6.. Then $A = \lim_{\to}(A_n, \phi_{n,n+1})$, $A_n = \mathbb{M}_{k_n}$, and $B = \lim_{\to}(B_n, \psi_{n,n+1})$, $B_n = \mathbb{M}_{k'_n}$, where $k_1 = k'_1 = 1$ and $k_{n+1} = \prod_{j=1}^n m_j$ and $k'_{n+1} = \prod_{j=1}^n m'_j$. Since m_1 divides $\boldsymbol{m}'$, it must divide k'_n for some n. Thus there is a unital embedding α_1 of A_2 into $B_{n'_1}$ for some $n'_1 > 2$. Similarly, $k'_{n'_1}$ divides k_{n_1} for some $n_1 > n'_1$, so there is a unital embedding β_1 of $B_{n'_1}$ into A_{n_1}. The maps $\beta_1 \circ \alpha_1$ and ϕ_{2,n_1} are unital embeddings of A_2 into A_{n_1}, and thus they are unitarily equivalent; by conjugating β_1 by this unitary, we may make $\beta_1 \circ \alpha_1 = \phi_{2,n_1}$. Now embed A_{n_1} into some $B_{n'_2}$ $(n'_2 > n_1)$ by α_2 with $\alpha_2 \circ \beta_1 = \psi_{n'_1,n'_2}$, and $B_{n'_2}$ into some A_{n_2} $(n_2 > n'_2)$ by β_2 with $\beta_2 \circ \alpha_2 = \phi_{n_1,n_2}$, and continue inductively. We thus get a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r] & A_2 \arrow[r] \arrow[dr, "\alpha_1"] & \cdots \arrow[r] & A_{n_1} \arrow[r] \arrow[dr, "\alpha_2"] & \cdots \arrow[r] & A_{n_2} \arrow[r] \arrow[dr, "\alpha_3\ \cdots"] & \cdots \arrow[r] & \cdots \arrow[r] & A \arrow[d, dotted, shift right, "\alpha"'] \\
B_1 \arrow[r] & \cdots \arrow[r] & B_{n'_1} \arrow[r] \arrow[ur, "\beta_1"] & \cdots \arrow[r] & B_{n'_2} \arrow[r] \arrow[ur, "\beta_2"] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & B \arrow[u, dotted, shift right, "\beta"']
\end{tikzcd}$$

and the downward and upward arrows induce mutually inverse isomorphisms between A_∞ and B_∞, hence between A and B. (We start at A_2 instead of A_1 since A_1 is just $\mathbb{C}$; we could have started at any A_n.)

I.3.1.10. Note that the proof shows a stronger result: if $\boldsymbol{m} = \boldsymbol{m}'$, then not only is $A \cong B$, but even the dense subalgebras A_∞ and B_∞ are isomorphic. This is special to UHF algebras (and, more generally, AF algebras), and is false for more complicated inductive systems ().

I.3.1.11. A similar proof shows that A embeds as a unital C*-subalgebra of B if and only if $\boldsymbol{m}$ "divides" $\boldsymbol{m}'$, i.e. the exponents e_p and e'_p satisfy $e_p \leq e'_p$ for every prime p. There is a "maximal" UHF algebra containing all UHF algebras as unital subalgebras: its supernatural number has all exponents ∞.

I.3.1.12. The supernatural numbers also exactly parametrize the noncyclic additive subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$ in the same manner, the group corresponding to the supernatural number $\boldsymbol{m}$ being the group of all rational numbers which can be written as fractions with a denominator dividing $\boldsymbol{m}$. So the UHF algebras can be exactly parametrized by the noncyclic additive subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$; the UHF algebra corresponding to a group G has a unique trace τ taking exactly the set $G \cap [0,1]$ of values on projections (and completely determining equivalence and subordination of projections). The group corresponding to the universal UHF algebra is $\mathbb{Q}$ itself. (If G is a cyclic subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$, then G must be generated by $\frac{1}{n}$ for some n, and the corresponding C*-algebra is just $\mathbb{M}_n$.)

I.3.1.13. The stable isomorphism classes of UHF algebras can also be described. $\mathbb{M}_{\boldsymbol{m}}$ and $\mathbb{M}_{\boldsymbol{m}'}$ are stably isomorphic if and only if there is a rational number t such that $\boldsymbol{m}' = t\boldsymbol{m}$, i.e. if e_p and e'_p are the exponents of the prime p for $\boldsymbol{m}$ and $\boldsymbol{m}'$ respectively, then e_p and e'_p differ for only finitely many p, and only by a finite amount for each p. In particular, $M_n(\mathbb{M}_{\boldsymbol{m}}) \cong \mathbb{M}_{n\boldsymbol{m}}$. For example, $M_2(\mathbb{M}_{2^\infty}) \cong \mathbb{M}_{2^\infty}$, but $M_3(\mathbb{M}_{2^\infty}) \cong \mathbb{M}_{2^\infty 3} \not\cong \mathbb{M}_{2^\infty}$.

Stable isomorphism can be described more cleanly in terms of the corresponding subgroups of $\mathbb{Q}$: two UHF algebras are stably isomorphic if and only if their corresponding groups are isomorphic.

I.3.1.14. Every UHF algebra can be written as an infinite tensor product of matrix algebras. If (m_n) is a sequence of integers > 1 with $\boldsymbol{m} = \prod_{n=1}^\infty m_n$, then $\mathbb{M}_{\boldsymbol{m}}$ is isomorphic to the infinite tensor product $\bigotimes_{n=1}^\infty \mathbb{M}_{m_n}$. Conversely, any infinite tensor product of matrix algebras is a UHF algebra. Any tensor product, finite or infinite, of UHF algebras is a UHF algebra; the corresponding supernatural number is the "product" of the supernatural numbers of the factors. The universal UHF algebra can be written $\bigotimes_{n=1}^\infty \mathbb{M}_n$ (or in many other ways as an infinite tensor product).

I.3.2. AF Algebras

The next step is to take inductive limits of finite-dimensional C*-algebras, i.e. direct sums of matrix algebras. There is no essential increased difficulty in going from single matrix algebras to direct sums except that the bookkeeping becomes much more complicated. The payoff is that we obtain many more examples, both nonsimple and (perhaps surprisingly) simple, of quite varied nature, with properties sometimes rather different from UHF algebras. And the complicated bookkeeping can ultimately be virtually avoided using K-theory.

We will primarily describe AF algebra theory for unital inductive limits. The general nonunital theory is essentially identical, but certain simplifications, largely notational, occur in the unital case.

AF algebras were named and first systematically studied by O. Bratteli [Bra72].

I.3.2.1. Definition. A C*-algebra A is a [unital] *AF algebra* if it is isomorphic to $\lim_{\to}(A_n, \phi_{n,n+1})$, where each A_n is finite-dimensional [and the $\phi_{n,n+1}$ are unital].

In other words, A is a (unital) AF algebra if it contains an increasing sequence of finite-dimensional (unital) C*-subalgebras whose union is dense. An AF algebra is by our definition separable (some authors extend the definition to include certain nonseparable C*-algebras too, but there is more than one inequivalent way to do this; cf. [Far19]). *AF* is supposed to stand for "approximately finite-dimensional" (cf. I.3.2.5.).

I.3.2.2. Examples. (i) Any finite-dimensional C*-algebra is an AF algebra.

(ii) Any UHF algebra is an AF algebra.

(iii) $\mathbb{K}$ is a nonunital AF algebra ().

(iv) If X is locally compact and second countable, then $C_0(X)$ is an AF algebra if and only if X is totally disconnected (zero-dimensional).

(v) Any quotient, finite direct sum, or finite tensor product of AF algebras is an AF algebra.

I.3.2.3. It is also true that an inductive limit of a sequence of AF algebras, hence an infinite tensor product of unital AF algebras, is AF, but the alternate characterization in I.3.2.5. is needed. It is harder to prove, but true, that an extension of AF algebras is AF ().

Local AF Algebras

There is a weaker sense in which a C*-algebra can be approximately finite-dimensional, reminiscent of the definition of an AFD von Neumann algebra:

I.3.2.4. Definition. Let A be a C*-algebra. Then A is an *AF algebra in the local sense* if for every finite subset F of A and $\epsilon > 0$ there is a finite-dimensional C*-subalgebra B of A such that for each $x \in F$ there is a $y \in B$ with $\|x - y\| < \epsilon$ (we say F is *ϵ-contained in B*, written $F \subseteq_\epsilon B$).

An AF algebra is obviously an AF algebra in the local sense. The converse is also true:

I.3.2.5. Proposition. Let A be a separable C*-algebra. Then A is an AF algebra if and only if it is an AF algebra in the local sense.

The proof () is based on the fact that a finite-dimensional C*-algebra D is semiprojective, and that two homomorphisms from D to a unital C*-algebra which are norm-close are conjugate via a unitary close to the identity.

Maps between Finite-Dimensional C*-Algebras

I.3.2.6. Let $A = \mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_r}$ be a finite-dimensional C*-algebra. As in I.3.1.5., a homomorphism ϕ from A to a matrix algebra $\mathbb{M}_n$ is necessarily (up to unitary equivalence) of the form

$$\phi(a_1, \dots, a_r) = diag(a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_r, \dots, a_r, 0, \dots, 0)$$

where a_j is repeated m_j times (we can have $m_j = 0$ for some j's if ϕ is not injective). The m_j are called the *multiplicities of the partial embeddings* of the summands of A into $\mathbb{M}_n$. If ϕ is unital, the final 0's are missing, and we necessarily have $m_1k_1 + \cdots + m_rk_r = n$.

I.3.2.7. So if $B = \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_s}$, any homomorphism ϕ from A to B must be of this form in each summand; thus ϕ is determined up to unitary equivalence by an $s \times r$ matrix of nonnegative integers (m_{ij}) giving the multiplicities of the partial embeddings of $\mathbb{M}_{k_j}$ into $\mathbb{M}_{n_i}$. The map ϕ is injective if no column of the matrix is 0. If ϕ is unital, then for each i we have $\sum_{j=1}^{r} m_{ij}k_j = n_i$, so no row of the matrix can be zero, and the n_i are completely determined by the k_j and the multiplicities m_{ij} (this is one of the notational simplifications in the unital case).

Bratteli Diagrams

I.3.2.8. So an inductive system of finite-dimensional C*-algebras can be completely described by specifying the number of summands at each stage, the size of each summand, and a matrix of multiplicities between each consecutive pair. Conversely, every such specification gives an inductive system; the only restriction is that the sizes of the matrix algebras at each stage must be large enough to absorb the embeddings. This data can be conveniently represented in a directed graph called a *Bratteli diagram*, where there is a sequence of rows, with the number of vertices in the n'th row the number of summands in the n'th algebra and the number of edges from a vertex in one row to a vertex in the next row equal to the multiplicity of the corresponding partial embedding. There must also be an integer at each vertex giving the size of the corresponding matrix algebra.

In the case of unital embeddings, the sizes of the matrix algebras in the first row completely determine the sizes at all subsequent rows, so the integers at the vertices can be omitted except at the beginning. We can even omit the beginning ones by always taking $A_1 = \mathbb{C}$, although it is often convenient not to do this. One other feature in the unital case is that every vertex in rows beyond the first has at least one edge from the previous row.

Although Bratteli diagrams are conventionally written with rows going down from the top, for printing ease and economy we will rotate 90° and draw them from left to right. (But we will still refer to "rows.")

For example, the standard Bratteli diagram for a UHF algebra has just one vertex in each row. For the CAR algebra we have

$$1 \;=\!=\!=\; 2 \;=\!=\!=\; 4 \;=\!=\!=\; 8 \;=\!=\!=\; \cdots$$

(although edges go from left to right, arrows are normally omitted), or for ease of drawing when multiplicities are large,

$$1 \xrightarrow{\;2\;} 2 \xrightarrow{\;2\;} 4 \xrightarrow{\;2\;} 8 \xrightarrow{\;2\;} \cdots$$

Since the embeddings are unital, the sizes can be omitted (with the convention that they are 1 at the initial vertices unless otherwise specified):

$$\cdot \;=\!=\!=\; \cdot \;=\!=\!=\; \cdot \;=\!=\!=\; \cdot \;=\!=\!=\; \cdots$$

or

$$\cdot \xrightarrow{\;2\;} \cdot \xrightarrow{\;2\;} \cdot \xrightarrow{\;2\;} \cdot \xrightarrow{\;2\;} \cdots$$

This is an example of a *stationary diagram*, where number of vertices and the (thus square) multiplicity matrices are the same at each step. Some, but not most, UHF algebras have stationary diagrams (exactly ones whose supernatural number is of the form n^∞ for some $n \in \mathbb{N}$).

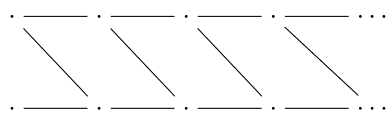
For a nonunital system, the sizes at the vertices must be specified. For example, a Bratteli diagram for $\mathbb{K}$ is

$$1 \;\text{———}\; 2 \;\text{———}\; 3 \;\text{———}\; 4 \;\text{———}\; \cdots$$

(cf. I.1.7.12.).

I.3.2.9. An AF algebra is not necessarily simple (in fact, "most" are not). There is an easy criterion characterizing when the algebra is simple, from II.1.12.4.: if $A = \lim_{\to}(A_n, \phi_{n,n+1})$ with each A_n finite-dimensional, then A is simple if and only if for each n there is an $n' > n$ such that each summand of A_n partially embeds in every summand of $A_{n'}$. In terms of the Bratteli diagram, this means that for every n there is an $n' > n$ such that for every vertex v in the nth row, there are paths from v to every vertex in the n'th row. A sufficient, but not necessary, condition insuring this is that at each stage every entry in the multiplicity matrix is positive.

An example of a Bratteli diagram giving a nonsimple limit is

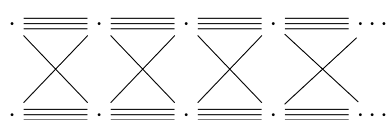


i.e. a stationary system with matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, whose inductive limit is $\mathbb{K} + \mathbb{C}1$ ().

I.3.2.10. Example. Consider the inductive limit A of the (unital) stationary system with matrix

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(this matrix was carefully chosen to make later calculations easy: it has a high degree of symmetry, and the size, row and column sums, and determinant are all powers of 2). The Bratteli diagram is



In more familiar terms, $A = \lim_{\to}(A_n, \phi_{n,n+1})$, where $A_n = \mathbb{M}_{4^{n-1}} \oplus \mathbb{M}_{4^{n-1}}$ and

$$\phi_{n,n+1}(a,b) = (diag(a,a,a,b), diag(a,b,b,b))$$

(where the entries are $4^{n-1} \times 4^{n-1}$ blocks).

By I.3.2.9., A is a simple unital AF algebra. Perhaps somewhat surprisingly, A has a unique trace. Each A_n has two extremal traces $\tau_{1,n}$ and $\tau_{2,n}$, which are the unique traces on the summands, and every trace on A_n is of the form $\lambda\tau_{1,n} + (1-\lambda)\tau_{2,n}$ for some $\lambda \in [0,1]$ (i.e. $\mathcal{T}(A_n) \cong [0,1]$). But the traces on A_{n+1}, when restricted to A_n, are just the ones with $\frac{1}{4} \leq \lambda \leq \frac{3}{4}$; iterating, the traces on A_{n+2}, restricted to A_n, are the ones with $\frac{3}{8} \leq \lambda \leq \frac{5}{8}$, etc. It is easy to check that the only trace on A_n which extends to a trace on A_{n+k} for all k is $\tau_n = \frac{1}{2}\tau_{1,n} + \frac{1}{2}\tau_{2,n}$. Thus there is a unique trace τ on A, and $\tau|_{A_n} = \tau_n$. (Actually the AF algebra defined by a stationary system always has unique trace.)

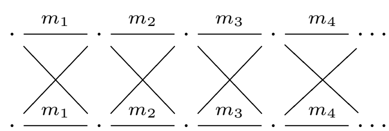
The values the trace τ takes on projections in A are exactly the dyadic rationals in $[0,1]$: since the summands of A_n have size a power of 2, $\tau_{1,n}$ and $\tau_{2,n}$ take dyadic rational values on projections, and hence so does τ_n since $\lambda = \frac{1}{2}$. It is easy to see that every dyadic rational in $[0,1]$ is attained in some A_n, hence in A. Thus the values are the same as the values of the trace on the CAR algebra. But there is a big difference between A and the CAR algebra: projections with the same trace need not be equivalent in A. For example, if $p = (1, 0)$ and $q = (0, 1)$ in $A_1 = \mathbb{C} \oplus \mathbb{C}$, then $\tau(p) = \tau(q) = \frac{1}{2}$, but p and q are not equivalent in A (if they were, they would be equivalent in A_n for some n by II.1.12.5., but this is impossible since they have different ranks in the summands of A_n, or alternatively there is a trace on A_n taking different values on p and q). Thus A is not a UHF algebra.

Although the trace τ on A does not determine equivalence, it does determine strict comparability: if p and q are projections in A which are not equivalent, then $p \precsim q$ if and only if $\tau(p) < \tau(q)$. In particular, if $\tau(p) = \tau(q)$, and r is a projection with $r \not\sim p$, $r \not\sim q$, then $r \precsim p$ if and only if $r \precsim q$.

I.3.2.11. Example. Consider the similar system where we vary the matrix from step to step, at the nth step taking

$$\begin{bmatrix} m_n & 1 \\ 1 & m_n \end{bmatrix}$$

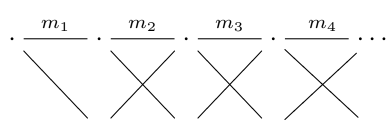
so the Bratteli diagram is



where the m_n are natural numbers. If A is the inductive limit, then A is a simple unital AF algebra. Each A_n has an interval of traces as before, and only a subinterval extend to A_{n+k} for any fixed k. However, if $m_n \to \infty$ rapidly enough, the intersection of these subintervals is a nontrivial interval for each n. Thus A has an entire interval of traces, i.e. $\mathcal{T}(A) \cong [0, 1]$. So a simple unital AF algebra need not have a unique trace. If τ_1 and τ_2 are the extremal traces, there are projections p and q with $\tau_1(p) < \tau_1(q)$ and $\tau_2(p) > \tau_2(q)$. It can be shown that the traces still determine strict comparison: if p and q are projections in A which are not equivalent, then $p \precsim q$ if and only if $\tau(p) < \tau(q)$ for every trace τ on A (in fact, this is true in every simple unital AF algebra ()).

By taking three summands at each stage, a simple unital AF algebra A can be constructed with $\mathcal{T}(A)$ a triangle; higher-dimensional simplexes can be obtained with more summands. By letting the number of summands grow, even an infinite-dimensional Choquet simplex (in fact, *any* metrizable Choquet simplex, cf. I.3.3.58.) can be obtained as the trace space of a simple unital AF algebra.

I.3.2.12. Example. One final example before moving on, with some number theory content (cf. [ES80]). Let θ be an irrational number in $[0, 1]$, with continued fraction expansion $(0; m_1, m_2, \dots)$. Consider the inductive system with nth multiplicity matrix $\begin{bmatrix} m_n & 1 \\ 1 & 0 \end{bmatrix}$, modified slightly at the first step to obtain the diagram



(note that this diagram is very different from the one in I.3.2.9.). Although the matrix entries are not all positive, the inductive limit A is nonetheless simple (one must just go two levels to get a full embedding).

It can be shown that A has a unique trace τ in a manner similar to I.3.2.10.. But the values τ takes on projections in A are precisely $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$ (), so in particular a simple unital AF algebra can have a unique trace which takes irrational values on projections. Write $\mathbb{M}_\theta$ for this AF algebra. We have $\mathbb{M}_{\theta'} \cong \mathbb{M}_\theta$ only if (in fact, if and only if) $\mathbb{Z} + \mathbb{Z}\theta' = \mathbb{Z} + \mathbb{Z}\theta$, which implies that $\theta' = \theta$ or $\theta' = 1 - \theta$ if $0 < \theta, \theta' < 1$, so there are $2^{\aleph_0}$ isomorphism classes of such algebras.

I.3.2.13. Although Bratteli diagrams give a nice combinatorial description of inductive systems and many properties of AF algebras can be read off from diagrams, they have two serious shortcomings as classification objects:

(i) The combinatorics can become oppressive.

(ii) More importantly, there is no canonical Bratteli diagram for an AF algebra, and there is no reasonable algorithm for determining when two Bratteli diagrams give isomorphic AF algebras (or for finding a Bratteli diagram for a C*-algebra not already constructed as an inductive limit of finite-dimensional algebras, but which is somehow known to be an AF algebra).

To illustrate (ii), it is a nice fairly simple exercise to show that the AF algebra given by the stationary system with matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is isomorphic to the CAR algebra.

Fortunately, there is a much nicer way to describe and classify AF algebras, using K-theory.

I.3.3. K_0 and Elliott's Classification of AF Algebras

The next piece of the story is K-theory. The development of operator algebra K-theory in the 1970s marked nothing short of a revolution in the subject, and was really the beginning of "noncommutative topology." In this chapter, we will only describe the rudiments of the theory in a rather disjointed and low-tech way (first K_0 and later K_1), emphasizing the connections with the classification program and somewhat following the history of the incorporation. For a more comprehensive and unified treatment, see [Bla98].

I.3.3.1. (Topological) K-theory began in [AH59], and by the 1960s was a part of topology (vector bundle theory). See e.g. [Ati67], [Kar08]. Algebraic K-theory, somewhat different from topological K-theory, followed (cf. [Bas68], [Ros94a]). The first explicit appearance of K-theory in operator algebras was in [BDF73]–[BDF77], although in retrospect it was implicit in the Murray-von Neumann type classification (I.2.2.), Glimm's classification of UHF algebras (I.3.1.9.), and in Fredholm index theory (K_1). K-Theory gradually became widely known and used in operator algebra theory in the late 1970s and 1980s; for example, the author, although already familiar with dimension groups, only really learned and understood K-theory in the summer of 1979 (four years after his Ph.D.) from a series of lectures by Larry Brown, who was undoubtedly the first operator algebra specialist to fully understand and appreciate the power and significance of K-theory. When the author began work on [Bla98] in 1982, there was still no readable treatment of the subject generally available except for [Tay75], which was useful but limited in scope. K-Theory is so central to the subject of C*-algebras today it is probably hard for younger specialists to imagine the dark ages before we had it.

K_0 for Unital and "Almost Unital" C*-Algebras

The following description of K_0 is valid on its face only for *unital* C*-algebras (and, more generally, for unital rings if "projection" is replaced by "idempotent"). K_0 of a nonunital C*-algebra must be defined in a different way (I.3.3.24.). With some additional work, this description is also valid for a C*-algebra A which is stably isomorphic to a unital C*-algebra (i.e. some matrix algebra over A contains a full projection), or if A is an inductive limit of unital C*-algebras, or equivalently if A has an approximate unit of projections (most generally, if $A \otimes \mathbb{K}$ has an approximate unit of projections), and in particular if A is an AF algebra.

I.3.3.2. A clue to the key idea is that for all the examples of simple unital C*-algebras with a unique trace considered so far, the values the trace takes on projections are of the form $G \cap [0,1]$, where G is an additive subgroup of $\mathbb{R}$. So we might want to manufacture a group from the equivalence classes of projections in the C*-algebra A.

The key idea is to not just consider projections in A, but also in matrix algebras over A. There are two reasons for doing this:

(i) Sometimes unexpected projections appear in matrix algebras which have nothing to do with projections in A itself. This turns out not to happen in most of the classifiable simple unital C*-algebras, but does occasionally occur (IV.3.5.3.), and it is very common in other C*-algebras, even (perhaps especially) in commutative C*-algebras.

(ii) We want to be able to add equivalence classes of projections ("orthogonal sum"), and we need the room matrix algebras provide to do this in general.

Considering matrix algebras is even more natural for commutative C*-algebras, where K-theory originated: if X is a compact Hausdorff space, the equivalence classes of projections in matrix algebras over $C(X)$ are in natural one-one correspondence with the isomorphism classes of complex vector bundles over X; orthogonal sum of projections corresponds to Whitney sum of the corresponding bundles.

I.3.3.3. Fix a unital C*-algebra A. If p is a projection in $M_n(A)$, identify p with the projection $diag(p,0)$ in $M_{n+m}(A)$ for any m (amplification). This gives an equivalence relation on $P = \cup_n \mathrm{Proj}(M_n(A))$. Consider the equivalence relation $\approx$ on P generated by this relation and equivalence in each $M_n(A)$, i.e. if p and q are projections in (possibly different) matrix algebras over A, then $p \approx q$ if there are amplifications p' and q' in a larger matrix algebra with $p' \sim q'$. Write $V(A)$ for the set of equivalence classes.

I.3.3.4. This equivalence relation can be described more cleanly by forming the "infinite matrix algebra" $M_\infty(A)$ as in I.1.7.12.. Then each projection in $M_n(A)$ has a canonical amplification to a projection in $M_\infty(A)$, and $\approx$ is just ordinary equivalence in $M_\infty(A)$ (which is not a C*-algebra but is a "local C*-algebra": any finite number of elements belong to a C*-subalgebra).

I.3.3.5. There is a natural binary operation on $V(A)$: $[p]+[q] = [diag(p,q)]$. It is not hard to see that this operation is well defined and makes $V(A)$ into an abelian semigroup with identity $[0]$ ().

I.3.3.6. Examples. (i) If $A = \mathbb{C}$, then $V(\mathbb{C}) \cong \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The same is true if $A = \mathbb{M}_n$ for any n.

(ii) If A is a UHF algebra corresponding to G, an additive subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ as in I.3.1.12., then $V(A) \cong G_+ = G \cap [0, \infty)$.

(iii) If $\mathbb{M}_\theta$ is as in I.3.2.12., then $V(\mathbb{M}_\theta) \cong (\mathbb{Z} + \mathbb{Z}\theta) \cap [0, \infty)$.

(iv) If $A = \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ separable and infinite-dimensional, it is easily seen that $V(A) \cong \mathbb{Z}_+ \cup \{\infty\}$.

(v) If $A = C(X)$, then $V(A)$ is isomorphic to the semigroup $V(X)$ of isomorphism classes of complex vector bundles over X with Whitney sum.

These examples suggest that $V(A)$ is an important isomorphism invariant of A.

I.3.3.7. If A and B are unital C*-algebras and $M_\infty(A) \cong M_\infty(B)$, then $V(A) \cong V(B)$. In particular, since $M_\infty(M_n(A)) \cong M_\infty(A)$ for any n, we have $V(M_n(A)) \cong V(A)$ for any n. More generally, if A and B are stably isomorphic, then $M_\infty(A) \cong M_\infty(B)$, so $V(A) \cong V(B)$.

I.3.3.8. The semigroup $V(A)$ contains much information about A, but abelian semigroups can be quite nasty: for example, $V(A)$ can fail to have cancellation (I.3.3.6.(iv); cancellation also often fails in $V(C(X))$ for X multidimensional). More subtle pathology is possible too (see e.g. ()).

It is much nicer to work with abelian groups, and much of the power of K-theory comes from passing to a group:

I.3.3.9. If S is an abelian semigroup, there is a universal enveloping group $G(S)$ for S, called the *Grothendieck group* of S. The Grothendieck construction is modeled on the construction of $\mathbb{Z}$ from $\mathbb{N}$. $G(S)$ can be thought of as a set of equivalence classes of formal differences of elements of S (if the operation in S is written additively), with $x_1 - y_1 \sim x_2 - y_2$ if there is a $z \in S$ with $x_1 + y_2 + z = x_2 + y_1 + z$ in S (the z is needed if S fails to have cancellation), with $(x_1 - y_1) + (x_2 - y_2) = (x_1 + x_2) - (y_1 + y_2)$ ($G(S)$ is, of course, also abelian). There is a natural homomorphism from S to $G(S)$, which is injective if and only if S has cancellation. The construction is functorial: if T is another abelian semigroup and $\phi : S \to T$ a semigroup homomorphism, there is an induced group homomorphism $\phi_G : G(S) \to G(T)$. The group $G(S)$ is universal: if ψ is a homomorphism from S to a group H, then ψ factors through $G(S)$. If S is already a group, then $G(S)$ is naturally isomorphic to S.

I.3.3.10. Definition. Let A be a unital C*-algebra. Then $K_0(A)$ is the Grothendieck group of $V(A)$.

I.3.3.11. If A is separable, then $V(A)$, and hence also $K_0(A)$, is countable (I.1.6.29.).

K_0 as an Ordered Group

I.3.3.12. We may lose a lot of information about A by passing from $V(A)$ to $K_0(A)$. But we can recover most of it (all except cancellation information) by putting a (pre)ordering on $K_0(A)$ by taking the image of $V(A)$ to be the positive cone $K_0^+(A)$. $K_0^+(A)$ thus denotes the set of elements of $K_0(A)$ which are represented by equivalence classes of projections, not just formal differences of such. If p and q are projections with $p \precsim q$, then $[p] \leq [q]$ in $K_0(A)$, but the converse is only "stably" true if $V(A)$ does not have cancellation. $K_0^+(A)$ always generates $K_0(A)$, i.e. every element of $K_0(A)$ is a difference of positive elements (in a highly nonunique way in general). But the ordering is only a preorder (transitive, reflexive relation) in general, not a partial order: $K_0^+(A) \cap [-K_0^+(A)]$ is not necessarily just $\{0\}$ unless A is stably finite (). Even when it is a partial order, it is generally not a total order.

I.3.3.13. Examples. (i) $(K_0(\mathbb{C}), K_0^+(\mathbb{C})) \cong (K_0(\mathbb{M}_n), K_0^+(\mathbb{M}_n)) \cong (\mathbb{Z}, \mathbb{Z}_+)$.

(ii) If A is a UHF algebra corresponding to G, an additive subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ as in I.3.1.12., then $(K_0(A), K_0^+(A)) \cong (G, G_+)$, where $G_+ = G \cap [0, \infty)$.

(iii) If $\mathbb{M}_\theta$ is as in I.3.2.12., then $(K_0(\mathbb{M}_\theta), K_0^+(\mathbb{M}_\theta)) \cong (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta)_+)$. Thus $K_0(\mathbb{M}_\theta) \cong \mathbb{Z}^2$ as a group, but the ordering is more delicate: for any θ and θ' we have $K_0(\mathbb{M}_\theta) \cong K_0(\mathbb{M}_{\theta'})$ as groups but (usually) not as ordered groups.

In (i)–(iii), $V(A)$ has cancellation and $(K_0(A), K_0^+(A))$ is a totally ordered group. Both are somewhat unusual, especially the second.

(iv) If $A = \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ infinite-dimensional, then $K_0(A) = \{0\}$. $V(A)$ does not have cancellation.

I.3.3.14. If $A = C(X)$ is commutative, the ordering on $K_0(A)$ has been used in topology to only a limited extent, since it is often pathological (it describes when one vector bundle is a subbundle of another). Considering the ordering on $K_0(A)$ is really an operator algebra phenomenon, which has turned out to be of fundamental importance there.

The Scale

I.3.3.15. There is one more item to add. The class $[1_A]$ is a positive element of $K_0(A)$, called the *scale* of $K_0(A)$. The triple $(K_0(A), K_0^+(A), [1_A])$ is called the *scaled ordered* K_0*-group* of A.

The element $[1_A]$ is not only positive, it is an *order unit*: For every $x \in K_0(A)$ there is an n such that $x \leq n[1_A]$ (note that $n[1_A] = [1_{M_n(A)}]$). We usually just write $[1]$ instead of $[1_A]$ if A is clearly understood.

I.3.3.16. The scale is what distinguishes A from $M_n(A)$ or, more generally, from C*-algebras stably isomorphic to A. If B is stably isomorphic to A, then $B \cong pM_n(A)p$ for some full projection p; then $(K_0(B), K_0^+(B)) \cong (K_0(A), K_0^+(A))$, but the scale of B is $[p]$, not $[1_A]$.

In fact, any order unit in $K_0(A)$ gives at least one C*-algebra stably isomorphic to A: if u is an order unit, then $u = [p]$ for some full projection p in a matrix algebra $M_n(A)$, and $(K_0(A), K_0^+(A), u) \cong (K_0(B), K_0^+(B), [1_B])$, where $B = pM_n(A)p$. But the p may not be unique up to equivalence if $V(A)$ fails to have cancellation, so there could be more than one C*-algebra stably isomorphic to A with the same scaled ordered K_0-group. (This turns out not to happen in classifiable simple unital C*-algebras, but this is a nontrivial result there.)

I.3.3.17. Examples. (i) $(K_0(\mathbb{M}_n), K_0^+(\mathbb{M}_n), [1]) \cong (\mathbb{Z}, \mathbb{Z}_+, n)$ for any n. Thus the scale distinguishes the algebras.

(ii) If A is a UHF algebra corresponding to G, an additive subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ as in I.3.1.12., then $(K_0(A), K_0^+(A), [1]) \cong (G, G_+, 1)$.

(iii) If $\mathbb{M}_\theta$ is as in I.3.2.12., then $(K_0(\mathbb{M}_\theta), K_0^+(\mathbb{M}_\theta), [1]) \cong (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta)_+, 1)$.

I.3.3.18. If A is a simple unital C*-algebra, then every nonzero positive element of $K_0(A)$ is an order unit (II.1.8.7.), so $(K_0(A), K_0^+(A))$ is a *simple ordered group*, i.e. has no nontrivial order ideals (hereditary subgroups).

K_0 for Factors

I.3.3.19. The K_0-theory of a factor can be easily described. If M is a countably decomposable factor (e.g. separable predual), then

If M is Type I_n (i.e. $M = \mathbb{M}_n$), then $V(M) \cong \mathbb{Z}_+$.

If M is Type I_∞, then $V(M) \cong \mathbb{Z}_+ \cup \{\infty\}$.

If M is Type II_1, then $V(M) \cong \mathbb{R}_+ = [0, \infty)$.

If M is Type II_∞, then $V(M) \cong [0, \infty]$.

If M is Type III, then $V(M) \cong \{0, \infty\}$.

If M is not countably decomposable, then $V(M)$ will have infinities of different sizes. So, whether or not M is countably decomposable, we have:

If M is Type I_n (i.e. $M = \mathbb{M}_n$), then $(K_0(M), K_0^+(M), [1_M]) \cong (\mathbb{Z}, \mathbb{Z}_+, n)$.

If M is Type II_1, then $(K_0(M), K_0^+(M), [1_M]) \cong (\mathbb{R}, \mathbb{R}_+, 1)$.

If M is properly infinite, then $K_0(M) = 0$.

Thus $K_0(M)$ enters into the classification of factors only in the most rudimentary way, only in the type classification, and not even in all of that; $V(M)$ is slightly more discriminating, giving the entire type classification but nothing more (for factors with separable predual). However, (ordered) K_0 is a nontrivial and essential part of any classification of separable simple C*-algebras.

States on Scaled Ordered Groups

I.3.3.20. If (G, G_+, u) and (H, H_+, v) are scaled preordered groups, i.e. G is an abelian group, G_+ is a subsemigroup of G which generates G, and u is an order unit in G_+, and similarly for H, a *homomorphism of scaled ordered groups* from (G, G_+, u) to (H, H_+, v) is a group homomorphism $\alpha : G \to H$ such that $\alpha(G_+) \subseteq H_+$ and $\alpha(u) = v$. An isomorphism of scaled ordered groups is defined similarly: α must be bijective and $\alpha(G_+) = H_+$, i.e. α is a group isomorphism and an order-isomorphism, preserving the scale.

I.3.3.21. Definition. Let (G, G_+, u) be a scaled preordered group. A *state* on (G, G_+, u) is a homomorphism σ from (G, G_+, u) to the scaled ordered group $(\mathbb{R}, \mathbb{R}_+, 1)$: σ is a group homomorphism from G to $\mathbb{R}$ which is order-preserving, i.e. $\sigma(x) \geq 0$ for $x \in G_+$, and $\sigma(u) = 1$. Let $\mathcal{S}(G, G_+, u)$ (often written $\mathcal{S}(G)$ if the ordering and scale are understood) be the set of states on (G, G_+, u), called the *state space* of (G, G_+, u).

I.3.3.22. $\mathcal{S}(G)$ is a compact convex set in the topology of elementwise convergence (states are bounded on any element since u is an order unit). $\mathcal{S}(G)$ can be empty if G is not partially ordered, but is nonempty if G is partially ordered (III.3.1.4.). If G is a simple ordered group, then $\mathcal{S}(G)$ "almost" determines the order on G (III.3.1.5.). If v is an order unit in G, then for every $\sigma \in \mathcal{S}(G, G_+, u)$, $\sigma(v) > 0$, and in fact the states are bounded below on v. In particular, if G is a simple ordered group, then every $\sigma \in \mathcal{S}(G)$ takes strictly positive values on $G_+ \setminus \{0\}$.

I.3.3.23. If A is a unital C*-algebra and τ is a trace on A, then τ defines a state on the scaled ordered K_0-group of A by $\sigma([p]) = \tau(p)$. This gives an affine map $\theta_A : \mathcal{T}(A) \to \mathcal{S}(K_0(A))$. For the simple unital C*-algebras considered so far (and for many more to come), θ_A is a bijection. The map θ_A is surjective in great generality (III.3.2.16., III.3.2.20.), but can fail to be injective even for classifiable simple C*-algebras.

K_0 of a Nonunital C*-Algebra

I.3.3.24. For completeness, we give the definition of K_0 of a nonunital C*-algebra A. There is a canonical *-homomorphism $\pi : A^\dagger \to \mathbb{C}$ with kernel A, and we define $K_0(A)$ to be the kernel of the corresponding homomorphism $K_0(\pi) : K_0(A^\dagger) \to K_0(\mathbb{C}) = \mathbb{Z}$, i.e. we have a (split) short exact sequence

$$0 \to K_0(A) \to K_0(A^\dagger) \to \mathbb{Z} \to 0$$

where the last map sends $[1_{A^\dagger}]$ to $1 \in \mathbb{Z}$. This is consistent with the unital definition: if A is unital, then $A^\dagger \cong A \oplus \mathbb{C}$, and π corresponds to projection onto the second coordinate. K-Theory for nonunital C*-algebras is discussed in more detail in III.5..

Functoriality and Inductive Limits

I.3.3.25. If A and B are unital C*-algebras and $\phi : A \to B$ is a unital homomorphism, then ϕ canonically defines a homomorphism from $M_\infty(A)$ to $M_\infty(B)$, also denoted ϕ. Since ϕ respects diagonal sums and equivalence of projections, there is an induced semigroup homomorphism $\phi_* : V(A) \to V(B)$, and hence a group homomorphism $\phi_* : K_0(A) \to K_0(B)$. This group homomorphism (sometimes denoted $K_0(\phi)$) sends $K_0^+(A)$ into $K_0^+(B)$, and $\phi_*([1_A]) = [1_B]$. Thus K_0 is (covariantly) functorial. [In topological K-theory, K^0 is contravariantly functorial; but $X \leftrightarrow C(X)$ is contravariantly functorial, so K_0 becomes covariant on the algebra level.]

Note, however, that if ϕ is injective [resp. surjective], ϕ_* need not be injective [resp. surjective] (although if ϕ is a bijection, so is ϕ_*).

I.3.3.26. If $(A_n, \phi_{n,n+1})$ is a unital inductive system of unital C*-algebras, then $(K_0(A_n), K_0(\phi_{n,n+1}))$ is an inductive system of abelian groups. We have easily that $K_0(A_\infty)$ is the algebraic direct limit

$$K_0(A_\infty) = \varinjlim (K_0(A_n), K_0(\phi_{n,n+1})) \ .$$

Since $K_0(\phi_{n,n+1})$ maps $K_0^+(A_n)$ into $K_0^+(A_{n+1})$, it follows that $K_0^+(A_\infty)$ is the union of the images of the $K_0^+(A_n)$, i.e. an element $x \in K_0(A_\infty)$ is positive if and only if it has a positive preimage in $K_0(A_n)$ for some n (for all sufficiently large n).

By II.1.12.7., we have that $(K_0(A), K_0^+(A), [1_A]) = (K_0(A_\infty), K_0^+(A_\infty), [1_{A_\infty}])$, so $(K_0(A), K_0^+(A), [1_A])$ is the algebraic direct limit of the $(K_0(A_n), K_0^+(A_n), [1_{A_n}])$ with respect to $K_0(\phi_{n,n+1})$ in the above sense.

The Scaled Ordered Group of an AF Algebra

I.3.3.27. Let $A = \mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_r}$ be a finite-dimensional C*-algebra. Then $K_0(A) \cong \mathbb{Z}^r$, with $K_0^+(A) = \mathbb{Z}_+^r$ the set of r-tuples with nonnegative entries (this is the *ordinary ordering* on $\mathbb{Z}^r$). The scale $[1_A]$ is $(k_1, \dots, k_r)$.

I.3.3.28. If $B = \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_s}$, and $\phi : A \to B$ is a unital homomorphism, with multiplicity matrix (m_{ij}), then ϕ_* is the homomorphism from $\mathbb{Z}^r$ to $\mathbb{Z}^s$ given by multiplication by the $s \times r$ matrix (m_{ij}). Conversely, if α is a homomorphism of scaled ordered groups from $(\mathbb{Z}^r, \mathbb{Z}^r_+, (k_1, \dots, k_r))$ to $(\mathbb{Z}^s, \mathbb{Z}^s_+, (n_1, \dots, n_s))$, then α is given by multiplication by an $s \times r$ matrix (m_{ij}) with integer entries. Since $\alpha(\mathbb{Z}^r_+) \subseteq \mathbb{Z}^s_+$, all the m_{ij} must be nonnegative, and since $\alpha(k_1, \dots, k_r) = (n_1, \dots, n_s)$, for each i we have $n_i = \sum_{j=1}^r m_{ij} k_j$. Thus (m_{ij}) is the multiplicity matrix for a unital homomorphism $\phi : A \to B$, i.e. $\alpha = \phi_*$; ϕ is unique up to unitary equivalence.

I.3.3.29. Thus if $A = \lim_{\to}(A_n, \phi_{n,n+1})$ is a unital AF algebra, we have that $(K_0(A), K_0^+(A), [1_A])$ is the direct limit of groups of the form $(\mathbb{Z}^{k_n}, \mathbb{Z}^{k_n}_+, \boldsymbol{v}_n)$, where $\boldsymbol{v}_n$ is a vector in $\mathbb{Z}^{k_n}_+$. Such an ordered group is called a *dimension group*; the [scaled] ordered K_0 group of A is called the *[scaled] dimension group of* A. Conversely, every scaled dimension group is the scaled ordered K_0 group of a unital AF algebra (not obviously unique at this point).

From the criterion in (), it follows easily that A is simple if and only if its dimension group is a simple ordered group.

Classification of AF Algebras

In [Ell76b], Elliott showed the following fundamental classification theorem:

I.3.3.30. Theorem. The scaled ordered dimension group is a complete isomorphism invariant for AF algebras: if A and B are AF algebras and $K_0(A) \cong K_0(B)$ as scaled ordered groups, then $A \cong B$.

I.3.3.31. The proof of this theorem is quite similar to the proof of I.3.1.9., using I.3.3.28., except that the bookkeeping is somewhat more complicated and a translation to the order structure is needed. See ().

I.3.3.32. This result applies in both the simple and nonsimple cases. It also applies to nonunital AF algebras, although there the scale is a little more complicated: instead of just a single order unit, it consists of all elements of $K_0(A)$ which come from a projection in A, an upward directed hereditary subset of $K_0^+(A)$ which generates $K_0(A)$ (in the unital case, the scale is really $\{x \in K_0(A) : 0 \leq x \leq [1_A]\}$).

I.3.3.33. Today we may regard this result as rather easy and fairly obvious. But it was far from obvious in the mid-1970s, before we knew K-theory, and [Ell76b] was considered a path-breaking paper which I think can be fairly regarded as the real beginning of the classification program. Actually, Elliott did not even realize he was doing K-theory when [Ell76b] was written (there is no mention of "K-theory" in the paper; the construction is clearly modeled on Murray-von Neumann and [Dix67]), although the connection was certainly made quickly thereafter. In fact, Elliott tells the story that Larry Brown said to him after seeing [Ell76b] that he was doing K_0. Elliott replied, "So?" and Brown said, "There's also K_1."

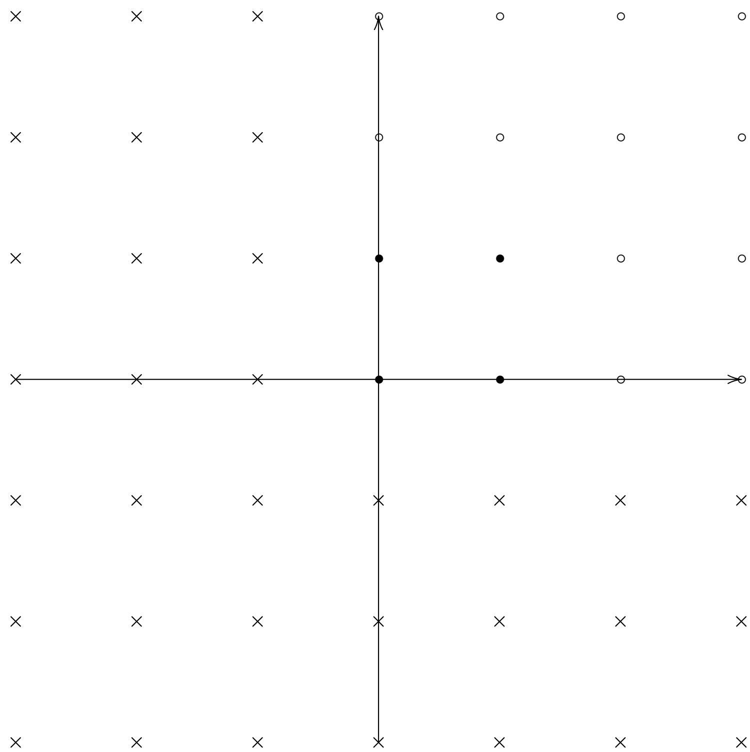
The proof of I.3.3.30. actually showed the following stronger result:

I.3.3.34. THEOREM. Let $A = \lim_{\to}(A_n, \phi_{n,n+1})$ and $B = \lim_{\to}(B_n, \psi_{n,n+1})$ be AF algebras with the A_n and B_n finite-dimensional, with algebraic direct limits A_∞ and B_∞. If α is a homomorphism of scaled ordered groups from $K_0(A)$ to $K_0(B)$, then there is a homomorphism $\phi : A \to B$ with $\phi_* = \alpha$ and $\phi(A_\infty) \subseteq B_\infty$. If α is an isomorphism, ϕ can be chosen to be an isomorphism with $\phi(A_\infty) = B_\infty$.

Thus every homomorphism of dimension groups comes from a homomorphism at the algebra level. In addition, the locally finite-dimensional subalgebra A_∞ of an AF algebra A is uniquely determined up to isomorphism. (This is quite special to locally finite-dimensional subalgebras of AF algebras; cf. ().)

Examples

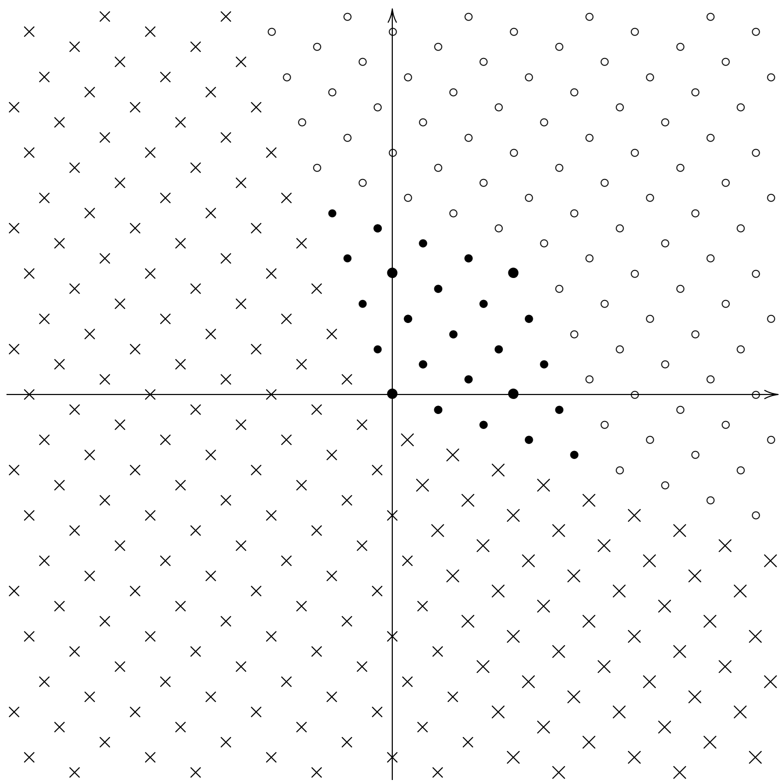
I.3.3.35. Let us examine the dimension group of the example in I.3.2.10.. The scaled ordered K_0 group of $A_1 = \mathbb{C} \oplus \mathbb{C}$ is, as a subset of $\mathbb{R}^2$, the following lattice:



where the dots indicate the positive cone and the solid dots the scale. This embeds into $K_0(A_2) \cong \mathbb{Z}^2$ via the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. We can instead pull back $K_0(A_2)$ to the same copy of $\mathbb{R}^2$ pictured above via the matrix

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

so the picture looks like



Repeating, $K_0(A_n)$ as a subset of $\mathbb{R}^2$ consists of a finer and finer lattice of dyadic rational points, with the positive cone an ever wider wedge and the scale a more and more elongated rhombus with two corners at $(0,0)$ and $(1,1)$. Thus, taking the union in this copy of $\mathbb{R}^2$, we get that $K_0(A)$ consists of all points in $\mathbb{R}^2$ with dyadic rational coordinates ($\mathbb{D}^2$), and since the rays from the origin through the two boundary edges of the positive cone of $K_0(A_n)$ both approach slope -1, the positive cone of $K_0(A)$ is the set of points strictly above the line $y = -x$, along with $(0,0)$. The order unit is $(1,1)$.

Applying the transformation $(x,y) \mapsto \left(\frac{x+y}{2}, \frac{y-x}{2}\right)$, this set becomes $\mathbb{D}^2$, with positive cone consisting of all points in the open right half-plane, along with $(0,0)$, i.e. $(x,y) \geq 0$ if and only if $x > 0$ or $x = y = 0$. This is the *strict ordering from the first coordinate* on $\mathbb{D}^2$. The order unit is now $(1,0)$. The equivalence classes of projections in A consist of $[0] = (0,0)$, $[1_A] = (1,0)$, and all $(x,y) \in \mathbb{D}^2$ with $0 < x < 1$.

I.3.3.36. **Exercise:** Do a similar analysis of the AF algebra with stationary Bratteli diagram with matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (and $A_1 = \mathbb{C}^2$). $K_0(A)$ might appear to be $\mathbb{D}_3 \oplus \mathbb{Z}$ with strict ordering from the first coordinate, where $\mathbb{D}_3$ is the triadic rationals, and order unit $(1,0)$, but this is not quite correct – it is a little more subtle (it is an index 2 subgroup).

I.3.3.37. A similar analysis can be done for the algebra of I.3.2.11.. One ends up with a dense subgroup of $\mathbb{Q}^2$ (the exact subgroup depends rather subtly on the m_n). But the rays through the edges of the positive cone approach two limiting rays R_1 and R_2 which do not lie on the same line. Applying a linear transformation of $\mathbb{R}^2$ sending points on the rays R_1 and R_2 to the positive x and y axes respectively, $K_0(A)$ becomes a dense additive subgroup G of $\mathbb{R}^2$ (it is not necessarily contained in $\mathbb{Q}^2$ since R_1 and R_2 can have irrational slope), with the positive cone consisting of the points in the open first quadrant, along with $(0,0)$. This is the *strict ordering* on G as a subset of $\mathbb{R}^2$. The order unit is a point of G in the open first quadrant; it can be arranged to be $(1,1)$ by scaling the axes appropriately.

I.3.3.38. There was somewhat of an industry in the late 1970s, continuing into the 1980s, studying special classes of dimension groups and their associated AF algebras using similar techniques, including totally ordered groups [Ell79], ultrasimplicially ordered groups [Han83], and symmetric dimension groups ([FM79], [FM81]). Some of this work is outlined in [Eff81] and in exercises in [Bla98, Chapter 7]; see also [**?**].

Characterization of Dimension Groups

To complete the classification of AF algebras, we need an intrinsic description of exactly which ordered groups are dimension groups. There is a clean and beautiful characterization, due to Effros, Handelman, and Shen [EHS80]. This material is covered in detail in [Goo86] (see also [Eff81]), so we give only a quick survey.

I.3.3.39. A few properties of dimension groups are immediately obvious. First, since an AF algebra (by our definition) is separable, dimension groups are countable. Second, a dimension group is a direct limit of free abelian groups, hence torsion-free. And since $\mathbb{Z}^r$ with the ordinary ordering is partially ordered and generated by its positive cone, the same is true of any dimension group.

A nonsimple, nonunital AF algebra need not contain a full projection, and thus a dimension group need not contain an order unit. Thus we will not work with order units in our analysis. So, for economy of language, we make the following definition:

I.3.3.40. Definition. An *ordered group* is a pair (G, G_+), where G is an abelian group and G_+ is a subsemigroup of G containing 0, such that G_+ generates G (i.e. $G_+ - G_+ = G$) and partially orders G (i.e. $G_+ \cap [-G_+] = \{0\}$).

A dimension group has an order property which is stronger than being torsion-free:

I.3.3.41. Definition. An ordered group (G, G_+) is *unperforated* if $x \in G$, $nx \in G_+$ (i.e. $nx \geq 0$) for some $n > 1$ implies $x \in G_+$ ($x \geq 0$). (G, G_+) is *weakly unperforated* if $x \in G$, $nx \in G_+ \setminus \{0\}$ (i.e. $nx > 0$) for some $n > 1$ implies $x \in G_+$ ($x > 0$).

Since $0 \in G_+$ and $G_+ \cap [-G_+] = \{0\}$, an unperforated group is torsion-free. A weakly unperforated group can have torsion, but is "unperforated up to torsion": any torsion element is "infinitesimal" (neither positive nor negative). An ordered group is unperforated if and only if it is weakly unperforated and torsion-free. The term "unperforated" means that the positive cone has no "holes." The generic example of an ordered group which is torsion-free but not unperforated is $\mathbb{Z}$ with positive cone $\{0\} \cup \{n : n \geq 2\}$.

I.3.3.42. A direct limit of (weakly) unperforated ordered groups is (weakly) unperforated. Since $\mathbb{Z}^r$ with the ordinary ordering is unperforated, so is any dimension group.

There is one more key property of dimension groups:

I.3.3.43. DEFINITION. An ordered group (G, G_+) has the *Riesz Interpolation Property* if, whenever $x_1, x_2, y_1, y_2 \in G$ and $x_1, x_2 \le y_1, y_2$ (i.e. $x_j \le y_k$ for $j, k = 1, 2$), then there is a $z \in G$ with $x_1, x_2 \le z \le y_1, y_2$.

I.3.3.44. The Riesz Interpolation Property is a weak lattice property. Any lattice-ordered group (in particular, any totally ordered group) has the Riesz Interpolation Property. Since $\mathbb{Z}^r$ with the ordinary ordering is lattice-ordered, it has the Riesz Interpolation Property.

I.3.3.45. For ordered groups, the Riesz Interpolation Property is equivalent to the *Riesz Decomposition Property*: if $y_1, y_2 \ge 0$ and $x \in G$ with $0 \le x \le y_1 + y_2$, then there are $x_1, x_2 \in G$ with $0 \le x_1 \le y_1$, $0 \le x_2 \le y_2$, and $x = x_1 + x_2$. Equivalently, if $x_1, \dots, x_n, y_1, \dots, y_m \in G_+$ and $x_1 + \cdots + x_n = y_1 + \cdots + y_m$, then there are $z_{jk} \in G_+$ with $x_j = \sum_{k=1}^m z_{jk}$ for all j and $y_k = \sum_{j=1}^n z_{jk}$ for all k. See e.g. [Goo86].

I.3.3.46. It is easy to see that a direct limit of ordered groups with the Riesz Interpolation Property also has the property. Since $K_0(A)$ has the Riesz Interpolation Property if A is finite-dimensional, every dimension group has the property too.

These properties characterize dimension groups:

I.3.3.47. THEOREM. [EFFROS-HANDELMAN-SHEN] Let (G, G_+) be an ordered group. Then (G, G_+) is a dimension group if and only if it is countable, unperforated, and has the Riesz Interpolation Property.

The proof of the theorem gives a recipe for writing such an ordered group (in a generally highly nonunique way) as a direct limit of groups of the form $\mathbb{Z}^r$ with ordinary ordering, which effectively gives a Bratteli diagram for the associated AF algebra, although in practice the procedure is not very reasonable to carry out in most specific cases (there appears to be no practical algorithm for generating a Bratteli diagram from a dimension group).

I.3.3.48. This gives a huge supply of dimension groups. For example, any countable totally ordered abelian group is a dimension group (this was proved in [Ell79] before [EHS80]). In particular, any countable additive subgroup G of $\mathbb{R}$ is a simple dimension group; so if G is normalized so that $1 \in G$, then $(G, G_+, 1)$ is order-isomorphic to the ordered K_0 group of a (unique) simple unital AF algebra $\mathbb{M}_G$. (This AF algebra has a unique trace and the values the trace takes on projections in $\mathbb{M}_G$ are precisely $G \cap [0, 1]$, and no two of these algebras can be isomorphic.) These algebras are a very special case of an even much larger supply of simple unital AF algebras (I.3.3.57.).

Choquet Simplexes

A Choquet simplex is a possibly infinite-dimensional generalization of an ordinary finite-dimensional simplex. We will consider compact convex sets in locally convex topological vector spaces, and we will restrict to metrizable compact convex sets whenever convenient, since all convex sets arising in the classification program (and, more generally, for separable C*-algebras) will be metrizable. An *isomorphism* of convex sets in topological vector spaces will mean an affine homeomorphism. See e.g. [Alf71] for a full treatment.

I.3.3.49. An *n-simplex* is a compact convex set Δ_n with exactly $n+1$ extreme points, such that every point in Δ_n is a unique convex combination of the extreme points. (It is called an n-simplex since it is topologically n-dimensional.) Realizations include the convex hull of the origin and the n standard basis vectors in $\mathbb{R}^n$, or the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$. We have:

A 0-simplex (Δ_0) is a single point.

A 1-simplex is a line segment, i.e. $\Delta_1 \cong [0,1]$.

A 2-simplex Δ_2 is a (solid) triangle.

A 3-simplex Δ_3 is a (solid) tetrahedron.

...

Most compact convex sets in Euclidean space are not simplexes. For example, a (solid) square is not a simplex since the center can be written as a convex combination of extreme points in two (actually many) different ways. A set like a disk is *a fortiori* not a simplex. The plural of *simplex* is often written *simplices*, which is correct Latin, but *simplexes* seems more common now (and is more consistent with the plural of *complex*).

I.3.3.50. Definition. Let K be a metrizable compact convex set in a locally convex topological vector space. K is a *(metrizable) Choquet simplex* if for each $x \in K$ there is a unique Borel probability measure supported on the extreme points of K whose barycenter is x.

A similar definition can be made in the nonmetrizable case, but is trickier in this case since the set of extreme points is not necessarily a Borel set. (Actually the standard definition of a Choquet simplex is that it is isomorphic to the base for a cone in a locally convex real topological vector space defining a lattice ordering. In the metrizable case this is equivalent to our definition, which is more useful for us.) A Choquet simplex in a finite-dimensional vector space is an n-simplex for some n. If compact convex sets are thought of as being defined by "generators" (the extreme points) and "relations," a Choquet simplex is a "free" compact convex set, although this should not be taken too literally unless the set of extreme points is closed.

I.3.3.51. If K is a compact convex set, write $\text{Aff}(K)$ for the set of real-valued continuous affine functions on K. Then $\text{Aff}(K)$ is a closed real subspace of $C(K)$ containing the real constant functions, and is thus a real Banach space under the supremum norm. Then K can be identified with a convex subset of the unit ball of $\text{Aff}(K)^*$ by point evaluation: $x \in K \mapsto \hat{x}$, where $\hat{x}(f) = f(x)$, and this identification is a homeomorphism using the weak-* topology on $\text{Aff}(K)^*$. This is the *standard representation* of K as a compact convex set in a locally convex topological vector space. $\text{Aff}(\Delta_n)$ can be identified with $\mathbb{R}^{n+1}$ with the ∞-metric (ℓ^∞_{n+1}), so $\text{Aff}(\Delta_n)^*$ is ℓ^1_{n+1}, with Δ_n the convex hull of the standard basis vectors.

There are two natural orderings on $\text{Aff}(K)$. One is the *ordinary ordering*, the usual ordering on functions, i.e. $f \leq g$ if $f(x) \leq g(x)$ for all $x \in K$. The positive cone consists of the nonnegative affine functions. The other ordering, more important for the classification, is the *strict ordering*, where $f \leq g$ if $f = g$ or $f(x) < g(x)$ for all $x \in K$; the positive cone consists of the constant function 0 and the functions which are strictly positive everywhere.

$\text{Aff}(K)$ is called $A(K)$ in many references, but we will stick with $\text{Aff}(K)$ since otherwise we would have to consider such confusing expressions as $A(\mathcal{T}(A))$.

The next result characterizing Choquet simplexes is nontrivial but standard, and suggests a connection between Choquet simplexes and dimension groups:

I.3.3.52. Theorem. Let K be a metrizable compact convex set in a locally convex topological vector space. Then the following are equivalent:

(i) K is a Choquet simplex.

(ii) $\text{Aff}(K)$ has the Riesz Interpolation Property in the strict ordering.

(iii) K can be written as an inverse limit of a sequence of finite-dimensional simplexes with surjective affine connecting maps.

Conditions (i) and (ii) are equivalent also in the nonmetrizable case.

I.3.3.53. There are a lot of infinite-dimensional Choquet simplexes. The most tractable class is the *Bauer simplexes*, Choquet simplexes whose set of extreme points is closed. These are obtained concretely as follows. Let X be a compact Hausdorff space. By the Riesz Representation Theorem, the dual of $C(X)$ is the space $M(X)$ of finite complex Radon measures on X. Let $\Delta_X = M_1^+(X)$ be the set of Radon probability measures on X (Borel probability measures if X is metrizable). Then Δ_X is a compact convex set in the weak-* topology (topology of pointwise convergence on $C(X)$). The extreme points of Δ_X are the point masses at points of X, a set homeomorphic to X (cf. I.1.6.6.(i)). That every point of Δ_X has a unique representing measure is essentially tautological. Thus Δ_X is a Bauer simplex. Every Bauer simplex arises in this manner. Δ_X is metrizable if (and only if) X is metrizable. A finite-dimensional simplex is a Bauer simplex: $\Delta_n = \Delta_X$, where X is a finite discrete space with $n+1$ points.

I.3.3.54. "Most" metrizable Choquet simplexes are not Bauer simplexes, and they are more complicated to describe. See () for a description of some. The most extreme example is the *Poulsen simplex* [Pou61], a metrizable Choquet simplex (unique up to isomorphism) whose set of extreme points is dense. Every metrizable Choquet simplex is isomorphic to a face of the Poulsen simplex.

There is an important extension of I.3.3.53. (which says that $\mathcal{T}(C(X)) \cong \Delta_X$):

I.3.3.55. THEOREM. Let A be a unital C*-algebra. If $\mathcal{T}(A) \neq \emptyset$, it is a Choquet simplex, which is metrizable if A is separable.

This result is due to E. THOMA [Tho64]. The proof uses von Neumann algebra theory and is beyond the scope of this book. Roughly speaking, if M is the finite part of the universal enveloping algebra A^{**}, there is a center-valued trace T on M, and every trace on M, and hence on A, is of the form $\phi \circ T$, where ϕ is a state on the center $Z(M)$. The state space of $Z(M)$ is a Bauer simplex, and $\mathcal{T}(A)$ is a nicely-behaved quotient of this simplex. For details see e.g. [Sak98, 3.1.18] or, for a similar argument, [BH82]; [BR24] contains a simplified proof. The fact that $\mathcal{T}(A)$ is metrizable if A is separable is elementary, since $\mathcal{S}(A)$ is metrizable in this case.

Simple Dimension Groups and Simple AF Algebras

I.3.3.56. If Δ is a metrizable Choquet simplex, then $\text{Aff}(\Delta)$ is separable. If G is a countable (uniformly) dense additive subgroup of $\text{Aff}(\Delta)$, then with the relative strict order G has the Riesz Interpolation Property (I.3.3.52.), hence is a simple dimension group. More generally, if G is a countable torsion-free abelian group and ρ is a homomorphism from G onto a dense subgroup of $\text{Aff}(\Delta)$, then G can be given the *strict order from* ρ, i.e. $g \in G_+$ if and only if $g = 0$ or $\rho(g)$ is strictly positive everywhere. Then (G, G_+) has the Riesz Interpolation Property and is thus a simple dimension group. The kernel of ρ is called the set of *infinitesimal elements* of G (they are neither positive nor negative).

If u is a fixed strictly positive element of G (order unit), then the picture can be modified so that $\rho(u)$ is the constant function 1 (although the simplex Δ can change in this process if it is not a Bauer simplex).

I.3.3.57. Conversely, if (G, G_+, u) is a scaled simple dimension group, then $\mathcal{S}(G)$ is a metrizable Choquet simplex (I.3.3.52.), and the representation of G on $\mathcal{S}(G)$ by point evaluation is a homomorphism ρ_G of G into $\text{Aff}(\mathcal{S}(G))$ with $\rho_G(u)$ the constant function 1; ρ_G is order-preserving, G has the strict order from ρ_G, and the range of ρ_G is dense () unless $G \cong \mathbb{Z}$. Thus every simple dimension group (other than $\mathbb{Z}$) is of this form. If A is the corresponding simple unital AF algebra, then $\mathcal{T}(A) \cong \mathcal{S}(G)$ and ρ_G is the evaluation map. In the example of I.3.3.35., the simplex is a single point, the group is $\mathbb{D}^2$, and ρ_G is projection onto the first coordinate. In the cases of UHF algebras, I.3.2.12., and more generally I.3.3.48., the simplex is a single point and ρ_G is an order-isomorphism. In the case of I.3.3.37., the simplex is $\Delta_1 \cong [0, 1]$.

We thus obtain the following characterization theorem for simple unital AF algebras. The invariant is fairly complicated, but clean and reasonably manageable.

I.3.3.58. THEOREM. The scaled ordered group $(K_0(A), K_0^+(A), [1_A])$ is a complete isomorphism invariant for a simple unital AF algebra A. The ordering on $K_0(A)$ is given by the evaluation map $\rho_A : K_0(A) \to \text{Aff}(\mathcal{T}(A))$: $K_0(A)$ has the strict ordering from ρ_A, i.e. $x \in K_0(A)$ is in $K_0^+(A)$ if and only if $x = 0$ or $\rho_A(x)$ is strictly positive. (Recall that $\mathcal{T}(A)$ is a Choquet simplex.) The range of ρ_A is (uniformly) dense in $\text{Aff}(\mathcal{T}(A))$ if A is infinite-dimensional. Conversely, every (G, u, Δ, ρ), where G is a countable torsion-free abelian group, $u \in G$, Δ a metrizable Choquet simplex, and ρ a homomorphism from G to $\text{Aff}(\Delta)$ with dense range, with $\rho(u)$ the constant function 1, arises in this manner by giving G the strict ordering from ρ: there is a unique simple unital AF algebra A with $(K_0(A), K_0^+(A), [1_A]) \cong (G, G_+, u)$. The correspondence is a bijection (of isomorphism classes).

I.3.3.59. This result is already a remarkable and significant achievement, and provides and classifies an enormous variety of simple unital C*-algebras. It also shows that the term "classification" must be interpreted in the proper way. In one sense, it is a clean and definitive description of all simple unital AF algebras in terms of an easily understood and (often) easily calculated invariant. However, the collection of invariants is pretty complicated, and arguably itself forms an "unclassifiable" set. For example, the classification of simple unital AF algebras entails, as a fairly small part, an effective description of all metrizable Choquet simplexes, which would include a classification of Bauer simplexes, equivalent to a classification of all compact metrizable spaces. On a much simpler level, the additive subgroups of $\mathbb{Q}^2$ containing $\mathbb{Z}^2$ are "unclassifiable" (in a precise technical sense, cf. [Tho03]), so if θ is a fixed irrational number, the additive subgroups of $\mathbb{Q} + \mathbb{Q}\theta$ containing $\mathbb{Z} + \mathbb{Z}\theta$ are unclassifiable. Each such G is a countable dense subgroup of $\mathbb{R}$ containing 1, so $(G, G_+, 1)$ is the scaled dimension group of a unique simple unital AF algebra (with unique trace, which determines equivalence and comparability), so even these relatively tame simple unital AF algebras are in a certain sense "unclassifiable." We will nonetheless regard I.3.3.58. as a true classification theorem (in the same sense as I.1.2.4.(ii)), which will be the prototype for later more general ones.

Classification of Simple Nonunital AF Algebras

I.3.3.60. Elliott's classification and the Effros-Handelman-Shen characterization of dimension groups hold equally in the nonunital case. We can also get a concrete picture as in I.3.3.58., but it is not quite canonical and there are some subtleties.

Perhaps the simplest way to obtain such a picture is to observe that if A is a simple nonunital AF algebra, then A contains many full (nonzero) projections. Pick one and let A_1 be the corresponding corner; then A is stably isomorphic to A_1 and can be realized as a hereditary C*-subalgebra of $A_1 \otimes \mathbb{K}$. (This realization is noncanonical in two ways: the choice of A_1 and the choice of a stable isomorphism.)

I.3.3.61. So we change point of view slightly and fix a simple unital AF algebra A_1, and describe nonunital C*-algebras stably isomorphic to A_1. This can be rather efficiently done by just considering upward directed hereditary generating subsets of the positive cone of $K_0(A_1)$, which amounts to simply considering strictly increasing sequences in the positive cone. The main subtlety (once A_1 is fixed) comes in identifying when two such subsets are conjugate under an order-automorphism of $K_0(A_1)$. The following AF examples illustrate the process.

I.3.3.62. Examples. (i) **Matroid C*-Algebras** [Dix67]. These are the nonunital C*-algebras stably isomorphic to UHF algebras. Fix a UHF algebra $\mathbb{M}_{\boldsymbol{m}}$, with supernatural number $\boldsymbol{m}$, and let G be the corresponding additive subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. If $\alpha \in \mathbb{R}$, $\alpha > 0$, set

$$S_\alpha = \{x \in G : 0 \leq x < \alpha\} \ .$$

Then S_α is an upward directed hereditary generating subset of G_+. In $\mathbb{M}_{\boldsymbol{m}} \otimes \mathbb{K}$, there is a strictly increasing sequence (p_n) of projections such that $\sup_n \tau(p_n) = \alpha$. Let $\mathbb{M}_{\boldsymbol{m},\alpha}$ be the completion of $\cup_n p_n(\mathbb{M}_{\boldsymbol{m}} \otimes \mathbb{K})p_n$. [$\mathbb{M}_{\boldsymbol{m},\alpha}$ is the inductive limit of the $p_n(\mathbb{M}_{\boldsymbol{m}} \otimes \mathbb{K})p_n$ under the inclusion maps.] $\mathbb{M}_{\boldsymbol{m},\alpha}$ is a nonunital AF algebra whose dimension group is (G, G_+, S_α), and in particular $\mathbb{M}_{\boldsymbol{m},\alpha}$ depends only on $\boldsymbol{m}$ and α and not on the choice of the p_n (this can be readily proved directly). We have that $\mathbb{M}_{\boldsymbol{m},\alpha} \cong \mathbb{M}_{\boldsymbol{m},\beta}$ if and only if the multiplication map by α/β from $\mathbb{R}$ to $\mathbb{R}$ sends G onto G (if $G = \mathbb{Q}$, this will happen if and only if $\alpha/\beta \in \mathbb{Q}$, but for other G the condition is more complicated and restrictive: we need $\alpha/\beta \in \mathbb{Q}$ and $\frac{\alpha}{\beta}\boldsymbol{m} = \boldsymbol{m}$; more generally, $\mathbb{M}_{\boldsymbol{m},\alpha} \cong \mathbb{M}_{\boldsymbol{n},\beta}$ if and only if $\alpha/\beta \in \mathbb{Q}$ and $\frac{\alpha}{\beta}\boldsymbol{m} = \boldsymbol{n}$). In particular, for fixed $\boldsymbol{m}$ there are $2^{\aleph_0}$

distinct $\mathbb{M}_{\boldsymbol{m},\alpha}$ up to isomorphism. (Note that even if $\alpha \in G$, $\mathbb{M}_{\boldsymbol{m},\alpha}$ is nonunital and hence not UHF.) C*-algebras of this form are called *matroid C*-algebras*. The $\mathbb{M}_{\boldsymbol{m},\alpha}$, although nonunital, have a (unique) trace, which is finite, and are algebraically simple. There is also one additional matroid C*-algebra associated to $\mathbb{M}_{\boldsymbol{m}}$, $\mathbb{M}_{\boldsymbol{m},\infty} = \mathbb{M}_{\boldsymbol{m}} \otimes \mathbb{K}$, which is not finite or algebraically simple (it has an unbounded trace). The matroid C*-algebras, like all AF algebras, are classified up to isomorphism by their scaled dimension groups.

(ii) Let A_1 be the simple unital AF algebra with dimension group $\mathbb{Q}^2$ with strict ordering and order unit $(1, 1)$ (actually any nonzero element of the positive cone gives an isomorphic algebra). There are three kinds of upward directed hereditary generating subsets of the positive cone without greatest elements, corresponding to nonunital C*-algebras stably isomorphic to A_1. One is the entire positive cone $S_{\infty,\infty}$, corresponding to $A_1 \otimes \mathbb{K}$. At the other extreme, if α and β are positive real numbers, the set

$$S_{\alpha,\beta} = \{(0, 0)\} \cup \{(x, y) \in \mathbb{Q}^2 : 0 < x < \alpha, 0 < y < \beta\}$$

corresponds to a nonunital C*-algebra $A_{\alpha,\beta}$ with two extremal tracial states. We have $A_{\alpha,\beta} \cong A_{\alpha',\beta'}$ if and only if α/α' and β/β' are rational (or if α/β' and β/α' are rational). The third kind is

$$S_{\alpha,\infty} = \{(0, 0)\} \cup \{(x, y) : 0 < x < \alpha, 0 < y\}$$

for $\alpha > 0$; these correspond to $A_{\alpha,\infty}$ which has one finite extremal trace and one infinite one. We have $A_{\alpha,\infty} \cong A_{\alpha',\infty}$ if and only if $\alpha/\alpha' \in \mathbb{Q}$. (There are also sets of the form $S_{\infty,\alpha}$ giving $A_{\infty,\alpha}$, but $A_{\infty,\alpha} \cong A_{\alpha,\infty}$.) The $A_{\alpha,\beta}$ are algebraically simple. The $A_{\alpha,\infty}$ are finite in the sense that they have a faithful tracial state, but are not algebraically simple since they also have an unbounded trace.

The group $\mathbb{Q}^2$ can be replaced by any dense subgroup containing $\mathbb{Z}^2$ (or, more generally, any countable dense subgroup of $\mathbb{R}^2$ containing $(1, 1)$); the corresponding scales and algebras are defined identically, but it is harder for two such algebras to be isomorphic (and the $A_{\infty,\beta}$ need not be isomorphic to any of the $A_{\alpha,\infty}$). In any case, for any such A_1 there are up to isomorphism $2^{\aleph_0}$ distinct nonunital C*-algebras of both types $A_{\alpha,\beta}$ and $A_{\alpha,\infty}$ stably isomorphic to A_1.

(iii) The situation is more complicated (and interesting!) if A_1 has infinitely many extremal traces. We take one relatively tame concrete example. Let X be the one-point compactification of $\mathbb{N}$, and Δ_X the corresponding Bauer simplex. Then $\mathrm{Aff}(\Delta_X)$ can be identified with the set of real-valued continuous functions on X, which can itself be identified with the set of convergent sequences of real numbers. To get the algebra A_1, we could take any countable dense group of convergent sequences containing the constant sequence 1 as dimension group, but to be specific let G be the set of eventually constant sequences of rational numbers. We will call the extreme points of Δ_X $\tau_1, \tau_2, \ldots, \tau_\omega$, since they correspond to the extremal traces on A_1.

If $x = (x_k)$ is any convergent sequence of strictly positive real numbers, with strictly positive limit, there is a strictly increasing sequence $g_n = (g_{n,k})$ of strictly positive eventually constant sequences of rational numbers converging uniformly to x, i.e. $(g_{n,k})$ is strictly increasing in n and $g_{n,k} \to x_k$ for each k. There is then an increasing sequence (p_n) of projections in $M_\infty(A_1)$ (actually in $M_m(A_1)$ for some m since x is bounded) corresponding to (g_n), i.e. $\tau_k(p_n) = g_{n,k}$ for all n and k and $\tau_\omega(p_n) = \lim_{k\to\infty} g_{n,k}$ for all n. Let A_x be the completion of $\cup_n p_n M_\infty(A_1) p_n$; then A_x is a nonunital AF algebra stably isomorphic to A_1 which depends (up to isomorphism) only on x. Since x is bounded, all traces on A_x are finite, and $T(A_x) \cong \Delta_X$ (this takes a bit of argument) and in particular $T(A_x)$ is compact. In this case, we can obtain a canonical representation by renormalization: identifying $\mathrm{Aff}(\Delta_X)$ with $C_\mathbb{R}(X)$, multiplication by $1/x$ gives a representation in which the scale just consists of 0 along with the elements of G strictly between 0 and 1 everywhere (note though that the group G may change under the renormalization).

There are many more nonunital AF algebras stably isomorphic to A_1 with all traces finite which are not of this form, however. Here are three interesting ones.

(a) Take a strictly increasing sequence (q_n) of projections in A_1 with $\tau_k(q_n) = k - \frac{1}{n}$ for $1 \leq k \leq n$ and $\tau_k(q_n) = 1 - \frac{1}{n}$ for $k > n$, $\tau_\omega(q_n) = 1 - \frac{1}{n}$. If A is the closure of $\cup_n q_n A_1 q_n$, then A is a simple nonunital AF algebra in which every trace is finite, but $T(A)$ is not compact: if $T_0(A)$ is the set of traces of norm ≤ 1, then $T_0(A)$ is a Choquet simplex with extreme points $\tau_1, \tau_2, \dots, \tau_\omega, 0$, and (τ_k) converges to 0. There does not appear to be a way to do a canonical normalization in this case.

(b) Take a strictly increasing sequence (r_n) of projections in A_1 with $\tau_k(r_n) = 1 - \frac{1}{n}$ for $1 \leq k \leq n$ and $\tau_k(r_n) = \frac{1}{2} - \frac{1}{n}$ for $k > n$, $\tau_\omega(r_n) = \frac{1}{2} - \frac{1}{n}$. If B is the closure of $\cup_n r_n A_1 r_n$, then B is a simple nonunital AF algebra in which every trace is finite, but $T(B)$ is not compact: if $T_0(B)$ is the set of traces of norm ≤ 1, then $T_0(B)$ is a Choquet simplex with extreme points $\tau_1, \tau_2, \dots, \tau_\omega, 0$, and (τ_k) converges to the midpoint of the segment between 0 and τ_ω. There is a reasonable canonical normalization in this case, namely the given one.

(c) Take a strictly increasing sequence (s_n) of projections in $M_2(A_1)$ with $\tau_k(s_n) = 1 - \frac{1}{n}$ for $1 \leq k \leq n$ odd, $\tau(s_n) = 2 - \frac{1}{n}$ for $1 \leq k \leq n$ even, and $\tau_k(s_n) = 1 - \frac{1}{n}$ for $k > n$, $\tau_\omega(s_n) = 1 - \frac{1}{n}$. If C is the closure of $\cup_n s_n M_2(A_1) s_n$, then C is a simple nonunital AF algebra in which every trace is finite, but $T(C)$ is not compact: if $T_0(B)$ is the set of traces of norm ≤ 1, then $T_0(B)$ is a Choquet simplex with extreme points $\tau_1, \tau_2, \dots, \tau_\omega, 0$, (τ_k), k odd, converge to τ_ω, and (τ_k), k even, converges to the midpoint of the segment between 0 and τ_ω. There is no obvious canonical normalization in this case.

There are also many nonunital algebras stably isomorphic to A_1 which have both finite and infinite traces. For example, let (t_n) be a strictly increasing sequence of projections in $M_\infty(A_1)$ with $\tau_k(t_n) = k - \frac{1}{n}$ for $1 \leq k \leq n$, $\tau_k(t_n) = n - \frac{1}{n}$ for $k > n$, $\tau_\omega(t_n) = n - \frac{1}{n}$. Then the completion D of $\cup_n t_n M_\infty(A_1) t_n$ has a sequence of finite extremal traces corresponding to the τ_n, and an unbounded extremal trace corresponding to τ_ω. $T(D)$ is again not compact. Variations can give any finite number of unbounded extremal traces and, if infinitely many, any finite or infinite number of bounded extremal traces.

(iv) More generally, let A_1 be any infinite-dimensional simple unital AF algebra, and let ρ be the evaluation map from $K_0(A_1)$ to $\mathrm{Aff}(\mathcal{T}(A_1))$ (recall that ρ has dense range). If f is any lower semicontinuous affine function from $\mathcal{T}(A_1)$ to $(0, +\infty]$, let

$$S_f = \{x \in K_0(A)_+ : [\rho(x)](\tau) < f(\tau) \text{ for all } \tau \in \mathcal{T}(A)\} \ .$$

Then S_f is an upward directed hereditary generating set in $K_0(A)_+$ without largest element, and thus there is a nonunital simple AF algebra A_f with scaled dimension group $(K_0(A), K_0(A)_+, S_f)$. A_f is stably isomorphic to A_1. A_f can be constructed concretely as follows. A strictly increasing sequence (p_n) of projections in $A_1 \otimes \mathbb{K}$ can be found with $\rho([p_n])$ converging pointwise to f; then A_f is isomorphic to the closure of $\cup_n p_n(A_1 \otimes \mathbb{K})p_n$. Every nonunital C*-algebra stably isomorphic to A_1 is of the form A_f for some such f, and as A_1 varies all nonunital simple AF algebras are obtained.

Do We Have Everything Yet?

I.3.3.63. We now have a large supply of separable simple C*-algebras, the AF algebras, and they are nicely classified. Do we have all separable simple C*-algebras yet, i.e. are they all AF algebras? The answer is no: for example, the reduced C*-algebras $C_r^*(\mathbb{F}_n)$ of free groups () are not nuclear [Tak64], and in particular, not AF algebras, so they are C*-analogs of non-AFD factors (and it was soon realized that there are impossibly many nonnuclear separable simple unital C*-algebras). But is every separable simple nuclear C*-algebra an AF algebra? No one, even in the mid-1970s, thought this was true: AF algebras seem much too special. And there were known examples of separable simple nuclear C*-algebras which seemed unlikely to be AF algebras, e.g. certain crossed products such as irrational rotation algebras (I.1.3.5.); there was also the possibility that there were purely infinite separable simple unital nuclear C*-algebras analogous to Type III factors, although there were no examples until [Cun77a]. The first simple unital nuclear C*-algebras proved to not be AF algebras were the Bunce-Deddens algebras (I.3.6.13.) [BD75]. The irrational rotation algebras seem far different from AF algebras since they are not even inductive limits in any obvious way. An early triumph of K-theory was a proof that the irrational rotation algebras are indeed not AF algebras; but they turn out to be much more similar to AF algebras than anyone imagined at the time.

I.3.4. Interlude: Existence and Abundance of Projections

Simple C*-algebras tend to have a lot of projections. Indeed, our mental picture of a simple unital C*-algebra is an infinite-dimensional analog of a matrix algebra. But there are in fact subtle questions about the existence and abundance of projections in simple C*-algebras. We will consider a few here; see III.1.1. for more information, and [Bla94] for a full discussion.

I.3.4.1. If A is a C*-algebra, not necessarily either simple or unital, we consider the following existence and abundance of projections properties A may or may not satisfy:

(∃P): A contains a nontrivial projection.

(LP): The linear span of the projections in A is dense in A.

(FS): The self-adjoint elements of finite spectrum are dense in A_{sa}.

The (FS) property is equivalent to two other interesting properties: (HP) and Real Rank 0 (RR0) (I.3.4.7., I.7.2.29.; cf. ()).

I.3.4.2. The (∃P) property is slightly different in the unital and nonunital cases: if A is unital it means A contains a projection different than 0 and 1, and in the nonunital case it just means A contains a nonzero projection. A C*-algebra which does not satisfy (∃P) is called *projectionless*, i.e. a unital C*-algebra is projectionless if it contains no projections except 0 and 1, and a nonunital C*-algebra is projectionless if it contains no nonzero projections. A is *stably projectionless* if its stable algebra $A \otimes \mathbb{K}$ is projectionless, i.e. every C*-algebra stably isomorphic to A is projectionless. A stably projectionless C*-algebra is necessarily nonunital.

I.3.4.3. It is obvious that (FS) ⇒ (LP) since a self-adjoint element with finite spectrum is a linear combination of mutually orthogonal projections. Since every element of a finite-dimensional C*-algebra has finite spectrum, any finite-dimensional C*-algebra satisfies (FS) and hence (LP). And clearly (LP) ⇒ (∃P) unless $A = \mathbb{C}$.

It is also clear that both (FS) and (LP) are preserved under inductive limits. Thus all AF algebras have (FS) (real rank 0), and hence also (LP).

If $A = C_0(X)$ is commutative, then A satisfies $(\exists P)$ if and only if X contains a nontrivial compact open subset; A is stably projectionless if and only if X contains no nonempty compact open subset, e.g. if X is connected and noncompact. The (LP) and (FS) properties of A are both equivalent to X being zero-dimensional (totally disconnected), which is also equivalent to A being an AF algebra.

I.3.4.4. In the primitive days (for operator algebras) of the 1960s and 1970s, it was regarded as a major open question whether every simple C*-algebra satisfies (∃P), i.e. whether projectionless simple C*-algebras exist (cf. [Kap58], [Dix96, 1.9.6], [Sak98, p. 242]). The first examples of projectionless simple C*-algebras were in [Bla80] (nonunital) and [Bla81] (unital); these were constructed in an *ad hoc* way using inductive limits similar to the ones in I.3.6.13. (the theory of dimension groups also played an essential role). Actually some projectionless simple unital C*-algebras were already known (the reduced C*-algebras of free groups), but they were not yet *known* to be projectionless. And, anyway, the $C_r^*(\mathbb{F}_n)$ are not nuclear, whereas the examples of [Bla80] and [Bla81] are, and are in fact in the classifiable class. (It was suspected that the $C_r^*(\mathbb{F}_n)$ were projectionless; but many people also thought the irrational rotation algebras were projectionless, and they actually turn out to have real rank 0, almost the opposite extreme! The faulty intuition about the irrational rotation algebras was not completely wrong, however: shortly after [Bla81] the crossed product of $C(S^3)$ by a minimal homeomorphism of S^3, a classifiable simple unital C*-algebra, was shown to be projectionless [Con81] ().)

[Bla81] also contains examples, both unital and nonunital, of simple C*-algebras which satisfy (∃P) but not (LP) (matrix algebras over suitable projectionless C*-algebras).

All these examples are separable.

Hereditary C*-Subalgebras

There are two more existence of projections properties we need to mention, which are based on hereditary subalgebras.

I.3.4.5. Definition. Let A be a C*-algebra. A C*-subalgebra B of A is *hereditary* if $a \in A$, $b \in B$, $0 \leq a \leq b$ implies $a \in B$. The hereditary C*-subalgebra B is *full* if it is not contained in any proper closed ideal of A.

Of course, every (nonzero) hereditary C*-subalgebra of a simple C*-algebra is full.

I.3.4.6. Examples. (i) If A is a C*-algebra and $x \in A$, then $x^*Ax = \{x^*ax : a \in A\}$ is a *-subalgebra of A, not closed in general, and its closure $Her_x(A) = \overline{x^*Ax}$ is a hereditary C*-subalgebra of A [Bla06, II.5.3.9]. If A is separable, every hereditary C*-subalgebra of A is of this form. In particular, every corner in A is a hereditary C*-subalgebra (corners are precisely the hereditary C*-subalgebras which are unital C*-algebras in their own right). The hereditary C*-subalgebra $Her_x(A)$ is full if and only if x is a full element, i.e. not contained in a proper closed ideal of A.

(ii) If L is a closed left ideal of A, then $L \cap L^*$ is a hereditary C*-subalgebra of A, and every hereditary C*-subalgebra is of this form [Bla06, II.5.3.2]. In particular, a closed (two-sided) ideal is a hereditary C*-subalgebra (never full for a proper ideal). The left ideal corresponding to the hereditary C*-subalgebra $Her_x(A) = \overline{x^*Ax}$ is $L = \overline{Ax}$.

I.3.4.7. If A is a C*-algebra, here are two more abundance of projections A may satisfy:

(SP): Every nonzero hereditary C*-subalgebra of A contains a nonzero projection.

(HP): Every hereditary C*-subalgebra of A has an approximate unit () of projections.

I.3.4.8. It is obvious that (HP) $\Rightarrow$ (SP). It is a nontrivial fact that (HP) is actually equivalent to (FS)–(RR0) (III.1.1.5.). Thus every AF algebra satisfies (HP), hence (SP) (in fact, every hereditary C*-subalgebra of an AF algebra is an AF algebra, although this is not obvious).

I.3.4.9. The (SP) property can be phrased without reference to hereditary C*-subalgebras: A has (SP) if and only if, whenever $a \in A_+$, $a \neq 0$, there is a nonzero projection $p \in A$ and $\lambda > 0$ such that $\lambda p \leq a$. Thus (SP) roughly means "A has arbitrarily small projections" (projections of arbitrarily small support).

Since any full hereditary C*-subalgebra of a C*-algebra A is Morita equivalent to A, if A is separable and simple A has (SP) if and only if no C*-algebra stably isomorphic to A is projectionless. Phrased in this way, (SP) is the "opposite extreme" to being stably projectionless.

If $A = C_0(X)$, then A has (SP) if and only if every nonempty open subset of X contains a nonempty compact open set. Thus (SP) is strictly weaker than (LP)–(HP)–(FS)–(RR0) in the commutative case: for example, if X has a dense set of isolated points, then $C_0(X)$ satisfies (SP).

I.3.4.10. The properties (SP) and (LP) are complementary: a C*-algebra, even a classifiable simple C*-algebra, may have either one without the other (or both, or neither). In fact, for classifiable simple unital C*-algebras, (SP) + (LP) is equivalent to (HP)–(FS)–(RR0).

I.3.5. AI Algebras

Instead of taking inductive limits of finite-dimensional C*-algebras, we might more generally consider inductive limits of "almost commutative" C*-algebras such as matrix algebras over commutative C*-algebras (or finite direct sums of such).

I.3.5.1. The first class we consider beyond AF algebras are the *AI algebras*, inductive limits of *interval algebras*: direct sums of algebras of the form $M_n(C(I))$, where $I = [0, 1]$. Since I is connected and thus $C(I)$ is projectionless, we might expect we can construct simple AI algebras which do not have enough projections to be AF algebras.

Since we do not require the connecting maps in an inductive limit to be injective, any C*-algebra which is an inductive limit of direct sums of quotients of matrix algebras over $C(I)$ is an AI algebra. In particular, since $\mathbb{C}$ is a quotient of $C(I)$, any AF algebra is an AI algebra. It turns out that any simple AI algebra can be written as an inductive limit of direct sums of matrix algebras over $C(I)$ with *injective* connecting maps (). (In fact, the discovery in [Bla90b] that AF algebras can have natural inductive systems of interval or circle algebras with injective connecting maps suggested that AF algebras are not as special as they appear and that classification beyond AF algebras, as had been proposed by EFFROS [**?**], might be possible; ELLIOTT took this idea and ran with it.)

I.3.5.2. Since $M_n(C(I)) \cong C(I, \mathbb{M}_n)$, a projection in $M_n(C(I))$ is just a projection-valued function from I to $\mathbb{M}_n$. By (), every such projection is equivalent to a constant function. So $V(C(I)) \cong V(\mathbb{C}) \cong \mathbb{Z}_+$, and

$$(K_0(M_n(C(I))), K_0^+(M_n(C(I))), [1]) \cong (K_0(\mathbb{M}_n), K_0^+(\mathbb{M}_n), [1]) \cong (\mathbb{Z}, \mathbb{Z}_+, n)$$

for any n. Thus from a K_0-theoretic standpoint matrix algebras over $C(I)$ behave just like finite-dimensional matrix algebras. (This follows more simply from the fact that $C(I)$ is homotopy equivalent to $\mathbb{C}$ and K-theory is homotopy invariant.)

It follows that the scaled ordered K_0-group of an AI algebra is a scaled dimension group. Thus AI algebras are K_0-theoretically indistinguishable from AF algebras.

I.3.5.3. The C*-algebra $C(I)$ has a lot of traces: any Borel probability measure on I gives a trace. Thus $\mathcal{T}(C(I))$ (and $\mathcal{T}(M_n(C(I)))$ for any n) is the Bauer simplex Δ_I (I.3.3.53.). However, any two traces agree on all projections: if τ is any trace on $M_n(C(I))$ and p is any projection in $M_n(C(I))$, then $\tau(p)$ is the (constant) normalized pointwise rank of p.

I.3.5.4. Example. We now give an elementary construction of a simple unital AI algebra which does not have (LP), hence cannot be an AF algebra. This is a prototype construction which has often been used in various forms (and which we will use repeatedly), and it illustrates how relatively easy it is to make a simple inductive limit. C*-Algebras constructed this way are called *Goodearl algebras* since the construction was first systematically examined in [Goo92].

We regard $M_n(C(I))$ as $C(I, \mathbb{M}_n)$. Choose a sequence (m_n) of natural numbers > 1 (we will want them to increase fairly rapidly) and set $k_1 = 1$ and $k_{n+1} = \prod_{j=1}^n m_j$ for each n. Let $A_n = C(I, \mathbb{M}_{k_n})$. Fix a dense sequence (t_n) in I. Embed A_n into $A_{n+1} \cong M_{m_n}(A_n)$ by

$$[\phi_{n,n+1}(f)](t) = diag(f(t), f(t), \ldots, f(t), f(t_n))$$

for $t \in I$, with $m_n - 1$ copies of the function f, and the last diagonal entry the constant function with value $f(t_n)$. Set $A = \lim_{\to}(A_n, \phi_{n,n+1})$.

We observe that A is simple no matter how the m_n (> 1) are chosen. Fix n. If J is a proper closed ideal of A_{n+r}, corresponding to the nonempty closed set X of I, then $J \cap \phi_{n,n+r}(A_n)$ is the ideal of A_n corresponding to the closed set $X \cup \{t_n, \ldots, t_{n+r-1}\}$. Thus if K is a proper closed ideal of A, then $K \cap \phi_n(A_n)$ is the ideal of A_n corresponding to a closed subset of I containing $\{t_k : k \geq n\}$. Since this set is dense in I, $K \cap \phi_n(A_n) = \{0\}$. This is true for all n, so $K = \{0\}$ (II.1.12.3.).

It is easy to calculate that the scaled ordered K_0-group of A is the same as the scaled dimension group of the UHF algebra corresponding to the supernatural number $\prod m_n$. In particular, all traces on A agree on projections.

However, if the m_n increase rapidly enough, A does not have unique trace. Fix an n. For each r, the fraction of the diagonal terms in $\phi_{n,n+r}$ which are the identity map is

$$\prod_{j=n}^{n+r-1} \frac{m_j - 1}{m_j} = \prod_{j=n}^{n+r-1} \left[1 - \frac{1}{m_j}\right]$$

so if these numbers are bounded away from 0, i.e. if the infinite product

$$\prod_{j=n}^{\infty} \left[1 - \frac{1}{m_j}\right]$$

converges, say to λ_n, $0 < \lambda_n < 1$, the trace on A_n corresponding to any measure on I of the form $\lambda_n \mu + \nu$, where μ is an arbitrary Borel probability measure on I and ν is a certain fixed purely atomic measure supported on $\{t_j : j \geq n\}$ of total mass $1 - \lambda_n$, extends to a trace on A. Thus $\mathcal{T}(A) \cong \Delta_I$. The infinite product converges if and only if

$$\sum_{n=1}^{\infty} \frac{1}{m_n} < \infty \ .$$

If $m_n = 2^n$ for all n, then A has an infinite-dimensional trace space and the same scaled ordered K_0 group as the CAR algebra. Since the traces on A all agree on the closed linear span of the projections of A, A cannot have (LP) and thus cannot be an AF algebra (the CAR algebra). By choosing other sequences (m_n), a similar example can be constructed corresponding to any supernatural number. (If (m_n) is bounded or increases too slowly to make the series converge, A is in fact a UHF algebra. **Good exercise:** show this.)

I.3.5.5. More generally, one could choose the m_n and A_n as above. For each n, fix continuous functions $g_{n,1}, \ldots, g_{n,m_n}$ from I to I, and define $\phi_{n,n+1}$ by

$$[\phi_{n,n+1}(f)](t) = diag(f(g_{n,1}(t)), \ldots, f(g_{n,m_n}(t))) \ .$$

Such an embedding is called a *diagonal embedding*. In the above example, $g_{n,j}(t) = t$ for $j < m_n$ and $g_{n,m_n}(t) = t_n$. The map $\phi_{n,n+1}$ is injective if and only if the ranges of the $g_{n,j}$ together cover I. If the $g_{n,j}$ sufficiently scramble I, the inductive limit is simple; the exact condition is complicated to describe but easily satisfied. Describing the trace space of the inductive limit is more of a challenge.

There are also more complicated connecting maps which are not diagonal embeddings, similar to the ones described in ().

Classification of Simple AI Algebras

I.3.5.6. In view of Example I.3.5.4., the scaled ordered K_0 group by itself cannot possibly be a complete isomorphism invariant for simple unital AI algebras; we must at least additionally specify something about the trace space. A way of doing this which turns out to suffice is using the map

$$\theta_A : \mathcal{T}(A) \to \mathcal{S}(K_0(A))$$

defined in I.3.3.23., i.e. the invariant will now become $(K_0(A), K_0^+(A), [1_A], \mathcal{T}(A), \theta_A)$. This was proved to be a complete invariant in [Ell93a]:

I.3.5.7. THEOREM. Let A and B be simple unital AI algebras. If

$$(K_0(A), K_0^+(A), [1_A], \mathcal{T}(A), \theta_A) \cong (K_0(B), K_0^+(B), [1_B], \mathcal{T}(B), \theta_B)$$

then $A \cong B$.

The proof follows the same outline as the proof of I.3.3.30., but is technically considerably more complex and difficult because embeddings are more complicated (I.3.5.5.) and the trace space must be taken into account, and it only works in the simple case. The isomorphism also does not respect the dense subalgebras A_∞ and B_∞ in general.

Approximate Intertwining

I.3.5.8. The key to adapting the proof of I.3.3.30. to the AI case, and farther, is a crucial observation due to ELLIOTT. To prove isomorphism of two algebraic direct limits $\lim_\to(A_n, \phi_{n,n+1})$ and $\lim_\to(B_n, \psi_{n,n+1})$, the standard approach is to find an intertwining

$$
\begin{tikzcd}
A_1 \arrow[r] & A_2 \arrow[r] \arrow[dr, "\alpha_1"] & \cdots \arrow[r] & A_{n_1} \arrow[r] \arrow[dr, "\alpha_2"] & \cdots \arrow[r] & A_{n_2} \arrow[r] \arrow[dr, "\alpha_3\ \cdots"] & \cdots \arrow[r] & \cdots \arrow[r] & A \arrow[d, dotted, shift left, "\beta"] \\
B_1 \arrow[r] & \cdots \arrow[r] & B_{n'_1} \arrow[r] \arrow[ur, "\beta_1"] & \cdots \arrow[r] & B_{n'_2} \arrow[r] \arrow[ur, "\beta_2"] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & B \arrow[u, dotted, shift left, "\alpha"]
\end{tikzcd}
$$

where the triangles exactly commute. However, in forming C*-inductive limits, the algebraic direct limit is completed. So to prove isomorphism of the inductive limits, it suffices to find such a diagram so that the triangles only *asymptotically* commute (see () for the technical requirement). Several versions of this principle have been critical in classification results, such as:

I.3.5.9. Theorem. [Elliott's Approximate Intertwining Argument] Let $(A_n, \phi_{n,n+1})$ and $(B, \psi_{n,n+1})$ be inductive systems of C*-algebras. If there is a diagram

$$
\begin{tikzcd}
A_1 \arrow[r] & A_2 \arrow[r] \arrow[dr, "\alpha_1"] & \cdots \arrow[r] & A_{n_1} \arrow[r] \arrow[dr, "\alpha_2"] & \cdots \arrow[r] & A_{n_2} \arrow[r] \arrow[dr, "\alpha_3\ \cdots"] & \cdots \arrow[r] & \cdots \arrow[r] & A \arrow[d, dotted, shift left, "\beta"] \\
B_1 \arrow[r] & \cdots \arrow[r] & B_{n'_1} \arrow[r] \arrow[ur, "\beta_1"] & \cdots \arrow[r] & B_{n'_2} \arrow[r] \arrow[ur, "\beta_2"] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & B \arrow[u, dotted, shift left, "\alpha"]
\end{tikzcd}
$$

in which the triangles commute (or even just asymptotically commute) up to approximate unitary equivalence (), then $\lim_{\to}(A_n, \phi_{n,n+1})$ and $\lim_{\to}(B_n, \psi_{n,n+1})$ are isomorphic.

In fact, more is true: the induced diagrams on K_0 and K_1 are exactly intertwined, thus give isomorphisms in K-theory, and the algebra isomorphism given by the theorem induces these isomorphisms on K-theory.

The proof of I.3.5.9. consists of modifying the α_n and β_n by composing them with inner automorphisms so that the modified diagram asymptotically commutes. See () for details.

I.3.5.10. The principle of I.3.5.8. can be extended: the diagonal maps only need to be asymptotically multiplicative. In fact, extending further, the connecting maps in the inductive systems themselves only need to be asymptotically multiplicative (indeed, even only asymptotically linear!) to give a well-defined inductive limit. This idea is exploited in [BK97].

Existence and Uniqueness Theorems

I.3.5.11. Using I.3.5.9., proofs of classification theorems such as I.3.5.7. are done in two parts:

> An *existence theorem*, that maps at the (ordered) K-theory and trace space level between building blocks can be realized by maps at the C*-algebra level.
>
> A *uniqueness theorem*, that the realizing map at the C*-algebra level is approximately unique.

In the AF case, the linear algebra theorem gives both existence and actual uniqueness up to unitary equivalence, but in the AI case and more general cases only approximate uniqueness is possible in general.

The Range of the Invariant

I.3.5.12. We of course also need to give an explicit description of the range the invariant can take, along the lines of I.3.3.58.. There are obvious restrictions: since the scaled ordered K_0-group of a simple unital AI algebra A is a simple dimension group (I.3.5.2.), it must be of the form of I.3.3.58., i.e. the state space must be a metrizable Choquet simplex Δ_s, and $K_0(A)$ must have the strict ordering from a homomorphism π from $K_0(A)$ onto a dense subgroup of $\mathrm{Aff}(\Delta_s)$ containing the constant function 1. We also have that $\mathcal{T}(A)$ is a metrizable Choquet simplex Δ_t, and θ is a continuous affine function from Δ_t to Δ_s. It is less obvious, but true, that θ must be surjective () and must map extreme points to extreme points [Tho94b].

It was shown in [Tho94b] that these are the only restrictions:

I.3.5.13. Theorem. Let $(G, u, \Delta_s, \pi, \Delta_t, \theta)$ consist of:

A countable torsion-free abelian group G;

An element $u \in G$ $(u \neq 0)$;

A metrizable Choquet simplex Δ_s;

A homomorphism ρ from G onto a dense additive subgroup of $\mathrm{Aff}(\Delta_s)$ with $\rho(u)$ the constant function 1 (so $G \ncong \mathbb{Z}$);

A metrizable Choquet simplex Δ_t;

A surjective continuous affine function $\theta : \Delta_t \to \Delta_s$ mapping extreme points to extreme points.

Give G the strict ordering from ρ and identify $\mathcal{S}(G, G_+, u)$ with Δ_s as in I.3.3.58.. Then there is a simple unital AI algebra A, unique up to isomorphism, with $(K_0(A), K_0^+(A), [1_A], \mathcal{T}(A), \theta_A) \cong (G, G_+, u, \Delta_t, \theta)$.

I.3.5.14. Comparing this result with I.3.3.58., and using the fact that every AF algebra is an AI algebra, we obtain:

I.3.5.15. Corollary. Let A be a simple unital AI algebra. The following are equivalent:

(i) A has (LP).

(ii) A has real rank zero.

(iii) The map $\theta_A : \mathcal{T}(A) \to \mathcal{S}(K_0(A))$ is injective (and hence an isomorphism).

(iv) A is an AF algebra.

Proof: We have already noted that (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i), and (i) $\Rightarrow$ (iii) since traces are linear and continuous. For (iii) $\Rightarrow$ (iv), if θ_A is injective, by I.3.3.58. there is a simple unital AF algebra B with the same invariant as A, and hence $A \cong B$ by I.3.5.7.. 👍

I.3.6. K_1 and $A\mathbb{T}$ Algebras

I.3.6.1. The next thing to try after AI algebras are $A\mathbb{T}$ *algebras*, inductive limits of direct sums of matrix algebras over $C(\mathbb{T})$, where $\mathbb{T} = S^1$ is a circle (*circle algebras*). These might seem like they shouldn't be too much harder or more general than AI algebras, but they are, and a new piece of the invariant is needed, the other part of K-theory.

K_1 of a C*-Algebra

I.3.6.2. Let A be a unital C*-algebra. The unitary group $\mathcal{U}(A)$ is a topological group with a large open connected normal subgroup $\mathcal{U}(A)_0$ (), so we may form the quotient group $\mathcal{U}(A)/\mathcal{U}(A)_0$. There is a deformation retraction from $A^{-1} = GL_1(A)$ to $\mathcal{U}(A)$ given by polar decomposition (), and it follows that $\mathcal{U}(A)/\mathcal{U}(A)_0 \cong GL_1(A)/GL_1(A)_0$. Since $\mathcal{U}(A)_0$ is an open subgroup, $\mathcal{U}(A)/\mathcal{U}(A)_0$ is countable if A is separable.

I.3.6.3. As with K_0, we can extend to matrix algebras. The map $u \mapsto diag(u, 1)$ is an injective homomorphism from the multiplicative group $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ into $\mathcal{U}_{n+1}(A) = \mathcal{U}(M_{n+1}(A))$ which sends $\mathcal{U}_n(A)_0$ into $\mathcal{U}_{n+1}(A)_0$, so drops to a homomorphism from $\mathcal{U}_n(A)/\mathcal{U}_n(A)_0$ to $\mathcal{U}_{n+1}(A)/\mathcal{U}_{n+1}(A)_0$. (The same embedding works identically for $GL_n(A)$.)

I.3.6.4. Definition. Let A be a unital C*-algebra. Then

$$K_1(A) = \lim_{\rightarrow} \mathcal{U}_n(A)/\mathcal{U}_n(A)_0 \ .$$

We could equally well work with $GL_n(A)/GL_n(A)_0$ in place of $\mathcal{U}_n(A)/\mathcal{U}_n(A)_0$. Note that $K_1(A)$ is countable if A is separable.

I.3.6.5. K_1 is functorial: if $\phi : A \to B$ is a unital homomorphism, ϕ gives a continuous group-homomorphism from $\mathcal{U}(A)$ to $\mathcal{U}(B)$ which maps $\mathcal{U}(A)_0$ into $\mathcal{U}(B)_0$. Since ϕ extends to a unital homomorphism from $M_n(A)$ to $M_n(B)$ for any n, we get a homomorphism $\phi_* : K_1(A) \to K_1(B)$, sometimes denoted $K_1(\phi)$.

I.3.6.6. The map $\mathcal{U}_n(A)/\mathcal{U}_n(A)_0 \to \mathcal{U}_{n+1}(A)/\mathcal{U}_{n+1}(A)_0$ is neither injective nor surjective in general. There is a natural homomorphism from $\mathcal{U}(A)/\mathcal{U}(A)_0$ to $K_1(A)$, which is *a fortiori* neither injective nor surjective in general. But if A is a classifiable simple C*-algebra, it turns out that this map is an isomorphism, so it is (after the fact) not necessary to pass to matrix algebras after all in this case (although this is indeed only after much fact).

I.3.6.7. The group $\mathcal{U}_n(A)/\mathcal{U}_n(A)_0$ is not abelian in general (even if A is commutative if $n > 1$!). But it is a fundamental fact that $K_1(A)$ *is* always abelian; the reason is that if u and v are unitaries in $M_n(A)$, while uv and vu need not be in the same component of $\mathcal{U}_n(A)$ (i.e. be homotopic), we have that $diag(uv, 1)$, $diag(u, v)$, and $diag(vu, 1)$ are all in the same component (i.e. are homotopic) in $\mathcal{U}_{2n}(A)$ ().

I.3.6.8. EXAMPLES. (i) The unitary group of any finite-dimensional C*-algebra is connected (i.e. $GL_n(\mathbb{C})$ is connected for any n). Thus $K_1(A) = 0$ for any finite-dimensional A (we write 0 for $\{0\}$).

(ii) More generally, K_1 of any von Neumann algebra is 0 by spectral theory.

(iii) A unitary in $C(I)$ is a continuous function from I to $\mathcal{U}(\mathbb{C}) \cong \mathbb{T}$. Since I is contractible, any such function is homotopic to a constant function. The same is true in matrix algebras, so $K_1(C(I)) = 0$.

(iv) A unitary in $C(\mathbb{T})$ is a continuous function from $\mathbb{T}$ to $\mathcal{U}(\mathbb{C}) \cong \mathbb{T}$. The group of homotopy classes is naturally isomorphic to $\pi_1(\mathbb{T}) \cong \mathbb{Z}$. It can be shown that the map from $\mathcal{U}(C(\mathbb{T}))/\mathcal{U}(C(\mathbb{T}))_0 \cong \mathbb{Z}$ to $K_1(C(\mathbb{T}))$ is an isomorphism (the determinant gives a cross section).

I.3.6.9. Because of (ii), K_1 does not arise in the classification of factors. And since K_1 respects inductive limits, i.e. if $A = \lim_{\to} A_n$, then $K_1(A)$ is the algebraic direct limit of $K_1(A_n)$ (II.1.12.7.), it follows from (i) and (iii) that K_1 of any AF or AI algebra is 0. This is why K_1 did not appear in the classification of AF or AI algebras.

However, by (iv) we might expect K_1 to enter in the theory of simple A$\mathbb{T}$ algebras. See I.3.6.12.–I.3.6.15. for some examples.

I.3.6.10. Note that K_0 is a purely algebraic construction: $K_0(R)$ can be defined in the same way for any unital ring R. But K_1 makes use of the topology, so the definition does not make sense for general unital rings. There is an algebraic K_1 defined for rings, but it does not coincide with K_1 as we have defined it for C*-algebras. Our definition of K_1, which generalizes the definition of topological K^1, is thus unique to C*-algebras (or at least to topological algebras somewhat similar to C*-algebras; it works fine for general Banach algebras).

I.3.6.11. There is far more to K-theory than we have discussed here. As we have defined them, there is no obvious connection between K_0 and K_1. But they are in fact intimately related: once they have been extended appropriately to nonunital C*-algebras, we have $K_1(A) \cong K_0(SA)$ for any A, where $SA = C_0(\mathbb{R}, A)$ is the "suspension" of A, and the deeper fact that $K_0(A) \cong K_1(SA)$ (*Bott periodicity*). There is a cyclic six-term exact sequence for extensions, and much more, all discussed in [Bla98]. Topological K-theory is an extraordinary cohomology theory, and for X a compact Hausdorff space we have (*Chern Character*):

$$K_0(C(X)) \otimes \mathbb{Q} \cong \bigoplus_{n=0}^{\infty} H^{2n}(X, \mathbb{Q})$$

$$K_1(C(X)) \otimes \mathbb{Q} \cong \bigoplus_{n=0}^{\infty} H^{2n+1}(X, \mathbb{Q})$$

i.e. up to torsion $K_0(C(X))$ is the even cohomology of X and K_1 the odd cohomology.

Simple A$\mathbb{T}$ Algebras

I.3.6.12. EXAMPLE. We can copy example I.3.5.4. essentially verbatim, using $\mathbb{T}$ in place of I. This time it is perhaps more interesting to take $m_n = 2$ for all n (in I.3.5.4. we just get the CAR algebra if we do that). In the same way as in I.3.5.4., we get a simple inductive limit, which has unique trace if the m_n do not increase rapidly and $\mathcal{T}(A) = \Delta_{\mathbb{T}}$ if they do.

However, we never get a UHF algebra or even an AI algebra. For we have $K_1(A_n) = \mathbb{Z}$ for all n, and it is easily seen that the map $K_1(\phi)$ is multiplication by $m_n - 1$. Thus, if each m_n is 2, we get that the inductive limit A has unique trace and the same scaled ordered K_0-group as the CAR algebra, but $K_1(A) = \mathbb{Z}$. If infinitely many m_n are greater than 2, we get that $K_1(A)$ is a noncyclic additive subgroup of $\mathbb{Q}$, not generally isomorphic to $K_0(A)$. The limit may have unique trace or many traces.

I.3.6.13. Example. Here is another example illustrating that embeddings are not necessarily diagonal embeddings and can be very complicated (in fact, this example does not properly show the complexity possible! See I.6.4.14.) We take $A_n = M_{2^n}(C(\mathbb{T})) \cong C(\mathbb{T}, \mathbb{M}_{2^n})$, and regard A_n as the set of continuous functions from $[0,1]$ to $\mathbb{M}_{2^n}$ for which $f(1) = f(0)$. As a first attempt, try a "diagonal" embedding $\phi_{n,n+1}$ defined by

$$[\phi_{n,n+1}(f)](t) = \begin{bmatrix} f\left(\frac{t}{2}\right) & 0 \\ 0 & f\left(\frac{t+1}{2}\right) \end{bmatrix}$$

i.e. the first diagonal entry has the values f takes from 0 to $\frac{1}{2}$, and the other diagonal entry has the values f takes from $\frac{1}{2}$ to 1. But the values at 0 and 1 are not quite the same, so this doesn't quite work; but since $f(1) = f(0)$, the values are the same except that the diagonal entries are reversed. This can be fixed since there is enough room in the matrix algebra to rotate. Thus we actually take

$$[\phi_{n,n+1}(f)](t) = \begin{bmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{bmatrix} \begin{bmatrix} f\left(\frac{t}{2}\right) & 0 \\ 0 & f\left(\frac{t+1}{2}\right) \end{bmatrix} \begin{bmatrix} \cos\frac{\pi t}{2} & \sin\frac{\pi t}{2} \\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{bmatrix}$$

(where all entries are $2^n \times 2^n$ blocks and the entries of the first and last matrix are scalar multiples of 1). The ends now match up. This is called a *standard twice-around embedding*. This may superficially appear to be similar to a two-fold covering of the circle by itself, but it is actually very different, essentially the opposite, and has no analog on the space level. One can similarly define an m-times around embedding of $C(\mathbb{T}, \mathbb{M}_n)$ into $C(\mathbb{T}, \mathbb{M}_{mn}) \cong M_m(C(\mathbb{T}, \mathbb{M}_n))$ for any m and n, conjugating by a path of unitaries from the identity to the cyclic shift unitary; multiplicities can be varied from step to step.

To see that the inductive limit is simple (in the twice-around case), a closed ideal J of A_n corresponds to a closed subset Z of $\mathbb{T}$, consisting of functions vanishing on Z. If J comes from a closed ideal of A_{n+1}, the set Z is invariant under rotation by π; if it comes from a closed ideal of A_{n+k}, Z is invariant under rotation by $2^{-k+1}\pi$. Thus if J comes from an ideal of the inductive limit, Z is invariant under a dense set of rotations and is therefore $\emptyset$ or $\mathbb{T}$. A similar argument works in the other cases.

These inductive limits are called *Bunce-Deddens algebras* since they were first described in [BD75]. The original description was in terms of weighted shifts, but it was soon noticed that they were of the form $C(\mathbb{T}) \rtimes_\alpha G$, where G is a dense torsion subgroup of $\mathbb{T}$ acting by translation (rotation). Such groups are classified by supernatural numbers in an obvious way. Such a G can be written as an increasing union $\cup G_n$ of finite cyclic subgroups G_n, so $C(\mathbb{T}) \rtimes_\alpha G \cong \varinjlim C(\mathbb{T}) \rtimes_\alpha G_n$. We have that $A_n = C(\mathbb{T}) \rtimes_\alpha G_n$ is isomorphic to $C(\mathbb{T}, \mathbb{M}_{k_n})$, where k_n is the order of G_n, and the embedding of A_n into A_{n+1} is a standard m_n-times around embedding with $m_n = \frac{k_{n+1}}{k_n}$.

The Bunce-Deddens algebras are classified (by their scaled ordered K_0-groups) exactly like UHF algebras. But $K_1(A) = \mathbb{Z}$ for any Bunce-Deddens algebra A.

I.3.6.14. It follows from the classification theorem () that the algebra constructed in I.3.6.12. with all $m_n = 2$ is isomorphic to the Bunce-Deddens algebra of type 2^∞. **Challenging exercise:** find an explicit isomorphism (the algebraic direct limits are not isomorphic!)

I.3.6.15. Example. We would be remiss in not mentioning another important example: the irrational rotation algebras A_θ (I.1.3.5.). These are not even expressible as inductive limits in any obvious way. However, a culmination of intense study by many authors ([Rie81], [Rie85], [AP89], [Put90]) was the remarkable demonstration in [EE93] that irrational rotation algebras are in fact A$\mathbb{T}$ algebras. It had already been shown in steps by various authors ([Rie81], [PV80a], [PV80b], [Rie83a], [CE90], [BKR92]) that the irrational rotation algebras have real rank 0 and the scaled ordered K_0 group of A_θ is $(\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta)_+, 1)$, and that $K_1(A_\theta) \cong \mathbb{Z}^2$. In fact, the study of irrational rotation algebras has motivated and catalyzed a lot of modern operator algebra theory, from aspects of K-theory and classification to noncommutative geometry and parts of mathematical physics.

Classification of Simple A$\mathbb{T}$ Algebras

The classification of simple unital A$\mathbb{T}$ algebras has the same form as for AI algebras, with K_1 thrown in independently [NT96]:

I.3.6.16. Theorem. Let $(G_0, u, \Delta_s, \pi, \Delta_t, \theta, G_1)$ consist of:

Countable torsion-free abelian groups G_0, G_1;

An element $u \in G_0$ ($u \neq 0$);

A metrizable Choquet simplex Δ_s;

A homomorphism π from G_0 onto a dense additive subgroup of $\mathrm{Aff}(\Delta_s)$ with $\pi(u)$ the constant function 1 (so $G_0 \ncong \mathbb{Z}$);

A metrizable Choquet simplex Δ_t;

A surjective continuous affine function $\theta : \Delta_t \to \Delta_s$ mapping extreme points to extreme points.

Give G_0 the strict ordering from π and identify $\mathcal{S}(G_0, G_0^+, u)$ with Δ_s as in I.3.3.58.. Then there is a simple unital A$\mathbb{T}$ algebra A, unique up to isomorphism, with $(K_0(A), K_0^+(A), [1_A], \mathcal{T}(A), \theta_A, K_1(A)) \cong (G_0, G_0^+, u, \Delta_t, \theta, G_1)$. Every simple unital A$\mathbb{T}$ algebra is of this form.

I.3.6.17. Comparing this result with I.3.5.13., and using the fact that every AI algebra is an A$\mathbb{T}$ algebra, we obtain:

I.3.6.18. Corollary. Let A be a simple unital A$\mathbb{T}$ algebra. Then A is an AI algebra if and only if $K_1(A) = 0$. A is an AF algebra if and only if $K_1(A) = 0$ and θ_A is a bijection, i.e. if and only if A has real rank zero and $K_1(A) = 0$.

The last statement was already known [EG96].

I.3.7. AH and ASH Algebras

We can continue in the same manner, working over increasingly complicated base spaces or even allowing the base spaces to vary. One might expect to get ever more complicated examples, and require ever more complicated invariants to classify them. We do get some more examples (involving torsion in K-theory), but it is remarkable that we do not need additional invariants, at least if we do not go too wild: the invariant which works for A$\mathbb{T}$ algebras is enough. We may even allow building blocks which are somewhat more general than matrix algebras over commutative C*-algebras, giving considerable additional examples but still not needing additional invariants.

Homogeneous C*-Algebras

I.3.7.1. Definition. Let $n \in \mathbb{N}$. A C*-algebra A is *n-homogeneous* if every irreducible representation of A has dimension exactly n. A is *homogeneous* if it is n-homogeneous for some n.

There are also $\aleph_0$-homogeneous C*-algebras, but they do not arise in the construction or classification of simple unital C*-algebras (although they do in the nonunital case). The base spaces are locally compact but not compact in general; we will for simplicity (in more than one sense ()!) stick to the case of compact base spaces (the unital case for n-homogeneous C*-algebras with $n \in \mathbb{N}$).

I.3.7.2. The 1-homogeneous C*-algebras are precisely the commutative C*-algebras. If A is n-homogeneous, then $X = \mathrm{Prim}(A)$ is Hausdorff and A is "locally" an $n \times n$ matrix algebra over $C(X)$ (more precisely, it is sections of a locally trivial bundle with base space X and fiber $\mathbb{M}_n$). The bundle need not be globally trivial, however: there are cohomological obstructions (of two types). The theory of these algebras is well understood, cf. ().

I.3.7.3. A corner in an n-homogeneous C*-algebra A over X is not necessarily quite homogeneous if X is not connected, but it is *locally homogeneous*: a finite direct sum of m-homogeneous C*-algebras for various $m \leq n$ over disjoint clopen subsets of X. If X is connected, any corner in an n-homogeneous C*-algebra over X is an m-homogeneous C*-algebra over X for some $m \leq n$. Conversely, a locally homogeneous C*-algebra $A = A_1 \oplus \cdots \oplus A_n$ (A_j homogeneous over X_j) is a corner in a homogeneous C*-algebra over the disjoint union of the X_j.

I.3.7.4. A full m-homogeneous corner in $M_n(C(X))$ is not necessarily isomorphic to $M_m(C(X))$, however: it could be the C*-algebra of sections of a nontrivial rank m complex vector bundle over X. Such a corner is stably isomorphic to $C(X)$.

There are even, for certain X and n, n-homogeneous C*-algebras over X which are not stably isomorphic to $C(X)$ (i.e. have nonzero Dixmier-Douady invariant). These have rarely been considered in the classification program.

AH Algebras

I.3.7.5. There are three classes of building blocks for inductive limits which may be, and have been, considered, of increasing generality:

(i) Direct sums of matrix algebras over commutative C*-algebras.

(ii) Locally homogeneous C*-algebras stably isomorphic to commutative C*-algebras (i.e. corners in matrix algebras over commutative C*-algebras).

(iii) General locally homogeneous C*-algebras (corners in homogeneous C*-algebras).

The name *AH algebra* has been used for inductive limits of all three classes, leading to some ambiguity of terminology. The classes become strictly larger as we go down, although it is unclear they are different for nicely behaved simple inductive limits. It is most common to use (ii) in the definition of AH algebras. For many base spaces, including intervals and circles, the three classes coincide so there is no ambiguity.

So we make the following definition:

I.3.7.6. Definition. An *AH algebra* is a C*-algebra which is an inductive limit of a sequence of C*-algebras of the form

$$\bigoplus_{j=1}^{r} p_j M_{n_j}(C(X_j))p_j$$

where, for each j, X_j is a compact metrizable space and p_j is a projection in $M_{n_j}(C(X_j))$ of uniform rank m_j.

Thus any A$\mathbb{T}$ algebra, and in particular any AF or AI algebra, is an AH algebra.

I.3.7.7. We would expect to get many simple unital AH algebras which are not A$\mathbb{T}$ algebras, and we do. For one thing, any $A\mathbb{T}$ algebra has torsion-free K-theory, but simple AH algebras over higher-dimensional base spaces can have torsion in K_0 and/or K_1. But if we exercise some self-control, that remarkably turns out to be all we get additionally.

I.3.7.8. However, there are some natural possibilities for the invariant which cannot be realized among simple AH algebras. For example, if B is a unital C*-algebra stably isomorphic to a commutative C*-algebra, then the state space of $K_0(B)$ is always a Choquet simplex. It follows easily () that if A is a unital AH algebra, the state space of $K_0(A)$ is a Choquet simplex. But although $\mathcal{T}(A)$ is always a Choquet simplex for any unital A, there is no reason the quotient $\mathcal{S}(K_0(A))$ should be a Choquet simplex in general, and it is not, even for classifiable simple C*-algebras (I.3.7.14.). There is a more subtle restriction also, even if $\mathcal{S}(K_0(A))$ is a Choquet simplex: if A is AH, the quotient map from $\mathcal{T}(A)$ to $\mathcal{S}(K_0(A))$ must send extreme points to extreme points ([Tho94b], [Vil98a]), but this is not necessarily true for classifiable algebras.

In addition, simple AH algebras tend to have a lot of projections: under mild hypotheses () they always have (SP). Thus if we hope to get simple C*-algebras with few or no projections, we must expand our class.

Subhomogeneous and ASH Algebras

I.3.7.9. Definition. Let $n \in \mathbb{N}$. A C*-algebra A is *n-subhomogeneous* if every irreducible representation of A has dimension $\leq n$. A is *subhomogeneous* if it is n-subhomogeneous for some n, i.e. if the dimensions of irreducible representations of A are bounded.

I.3.7.10. Any C*-subalgebra of an n-homogeneous C*-algebra is n-subhomogeneous. Conversely, if A is n-subhomogeneous, then A is isomorphic to a C*-subalgebra of $M_m(C(X))$ for some X and m, which may be somewhat larger than n [Bla06, IV.1.4.3].

I.3.7.11. Subhomogeneous C*-algebras can be much more complicated than homogeneous C*-algebras, and the primitive ideal space need not be Hausdorff (it is always T_1). The examples of () are 2-subhomogeneous. Here are two variations on these examples which will be important.

I.3.7.12. Example. Let A be the C*-subalgebra of $C([0,1],\mathbb{M}_2)$ consisting of all continuous functions f for which $f(0)$ and $f(1)$ are diagonal matrices. Then A is 2-subhomogeneous and $\text{Prim}(A)$ consists of $[0,1]$ with 0 and 1 doubled, thus is not Hausdorff.

There are four equivalence classes of rank 1 projections in A: $[p_{11}]$, where p_{11} is the constant function with value $diag(1,0)$; $[p_{22}]$, where p_{22} is the constant function $diag(0,1)$; and $[p_{12}]$ and $[p_{21}]$, where

$$p_{12}(t)=\begin{bmatrix}\cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2}\\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\cos\frac{\pi t}{2} & \sin\frac{\pi t}{2}\\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}\end{bmatrix}$$

$$p_{21}(t)=\begin{bmatrix}\cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2}\\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\cos\frac{\pi t}{2} & \sin\frac{\pi t}{2}\\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}\end{bmatrix}$$

and no two of these are equivalent. However, $p_{11}+p_{22}=p_{12}+p_{21}=1$. It can be shown that $K_0(A)\cong\mathbb{Z}^4/\langle(1,-1,1,-1)\rangle\cong\mathbb{Z}^3$, with positive cone generated by the equivalence classes of the four projections. There are four extremal states $\sigma_{0,1},\sigma_{0,2},\sigma_{1,1},\sigma_{1,2}$ on $K_0(A)$, taking values

$$\sigma_{0,1}(p_{11})=\sigma_{0,1}(p_{12})=1,\quad \sigma_{0,1}(p_{21})=\sigma_{0,1}(p_{22})=0$$

$$\sigma_{0,2}(p_{21})=\sigma_{0,2}(p_{22})=1,\quad \sigma_{0,2}(p_{11})=\sigma_{0,2}(p_{12})=0$$

$$\sigma_{1,1}(p_{11})=\sigma_{1,1}(p_{21})=1,\quad \sigma_{1,1}(p_{12})=\sigma_{1,1}(p_{22})=0$$

$$\sigma_{1,2}(p_{12})=\sigma_{1,2}(p_{22})=1,\quad \sigma_{1,2}(p_{11})=\sigma_{1,2}(p_{21})=0$$

but $\sigma_{0,1}+\sigma_{0,2}=\sigma_{1,1}+\sigma_{1,2}$. Thus $\mathcal{S}(K_0(A))$ is a square. We have $K_1(A)=0$.

I.3.7.13. Example. Fix $p,q\in\mathbb{N}$. Consider the C*-subalgebra of $C([0,1],\mathbb{M}_p\otimes\mathbb{M}_q)\cong C([0,1],\mathbb{M}_{pq})$ defined by

$$I_{p,q}=\{f\in C([0,1],\mathbb{M}_p\otimes\mathbb{M}_q): f(0)\in\mathbb{M}_p\otimes 1, f(1)\in 1\otimes\mathbb{M}_q\}\ .$$

This C*-algebra is called the *dimension drop algebra* of size (p,q). It is unital and pq-subhomogeneous. Note that $I_{1,1}=C([0,1])$. If p and q are relatively prime, it is projectionless. We have $K_0(I_{p,q})\cong\mathbb{Z}$, generated by $[1_A]$ if p and q are relatively prime. And $K_1(A)\cong\mathbb{Z}_d(=\mathbb{Z}/d\mathbb{Z})$, where $d=gcd(p,q)$ (so if p and q are relatively prime, $K_1(I_{p,q})=0$).

Simple ASH Algebras

I.3.7.14. Example. We begin with I.3.7.12. and apply the construction of I.3.5.4.. We must just exercise slight notational care. We choose a rapidly increasing sequence (m_n) as before, with each $m_n > 2$, and a dense sequence (t_n) in $(0, 1)$. Take A_1 to be the C*-algebra of I.3.5.4., and at the nth stage embed into the $m_n \times m_n$ matrices over the previous algebra. We explicitly describe the first embedding. We may regard $M_{m_1}(A_1)$ as the continuous functions from $[0, 1]$ to $\mathbb{M}_{2m_1} \cong M_{m_1}(C([0, 1], \mathbb{M}_2))$ consisting of continuous functions having the property that the values at 0 and 1 consist of an $m_1 \times m_1$ matrix of diagonal 2×2 matrices. Then by conjugating by the matrix $diag(1, \dots , 1, u)$, where u is the 4×4 matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we may assume the last four rows and columns are 4×4 blocks which at 0 and 1 are block diagonal.

If $f \in A_1$, define $\phi_{1,2}(f)$ by

$$[\phi_{1,2}(f)](t) = diag(f(t), \dots , f(t), f(t_n), f(t_n))$$

where we take $m_1 - 2$ copies of $f(t)$. It is easy to check that $\phi_{1,2}(f)$ takes values in $M_{m_1}(A_1)$ as described above. The succeeding embeddings are defined analogously. Let A be the inductive limit.

As in I.3.5.4., A is simple no matter how the m_n are chosen. If the m_n grow sufficiently rapidly, the state space of $K_0(A)$ is still a square. Thus A is a simple unital ASH algebra, but cannot be AH (I.3.7.8.).

This example, or at least one very similar, is due to Elliott (cf. [ET94]).

I.3.7.15. Example. Here is a more complicated and subtle example. We omit some technical details.

We first define some functions from $[0, 1]$ to $[0, 1]$. Let (q_n) be an enumeration of $\mathbb{Q} \cap [0, 1]$. Define

$$\phi_n(t) = q_n t$$

$$\psi_n(t) = \begin{cases} t/q_n & \text{if} \quad 0 \le t \le q_n \\ 1 & \text{if} \quad q_n < t \le 1 \end{cases}$$

$$\theta_n(t) = \begin{cases} nt & \text{if} \quad 0 \le t \le 1/n \\ 1 & \text{if} \quad 1/n < t \le 1 \end{cases}$$

(note that $\theta_n = \psi_m$, where $q_m = 1/n$). We have that $\phi_n(0) = \psi_n(0) = \theta_n(0) = 0$ for all n and

$$(\psi_k \circ \phi_n)(t) = \begin{cases} q_n t/q_k & \text{if} \quad 0 \le t \le q_k \\ q_n & \text{if} \quad q_k < t \le 1 \end{cases}$$

for each n and k with $q_k < q_n$, and it follows that for any $t > 0$, the images of t under compositions of the ϕ_n and ψ_n are dense in $(0, 1]$. This will insure that the inductive limit we will construct is simple.

Now let

$$A_1 = \{f \in C([0, 1], \mathbb{M}_2) : f(0) \text{ diagonal}\} .$$

Then $K_0(A_1) \cong \mathbb{Z}^2$. Successively embed matrix algebras over A_1 into larger matrix algebras over A_1 by

$$f \mapsto u^* diag(f \circ \phi_n, f \circ \psi_n, f \circ \theta_n, \ldots, f \circ \theta_n, f(0), \ldots, f(0))u$$

where $f(0)$ denotes the constant function with value $f(0)$, $f \circ \theta_n$ and $f(0)$ are repeated m_n times, and u is a permutation matrix to make the value lie in a matrix algebra over A_1.

Let $A = \lim\limits_{\to} A_n$. Then A is a simple unital ASH algebra, and $K_0(A)$ has rank 2 (the exact group depends on the choice of the m_n). If $m_n \to \infty$ rapidly enough, A has exactly three extremal traces τ_{01}, τ_{02}, and τ_1 corresponding to the end traces on A_1, i.e. $\mathcal{T}(A)$ is a triangle with extreme points τ_{01}, τ_{01}, and τ_1. The states on $K_0(A)$ from τ_{01} and τ_{02} are the extremal states, but the state from τ_2 is the average of the extremal states (i.e. corresponds to the point $1/2$ of $[0,1]$). Thus, even though the state space of $K_0(A)$ is the simplex $[0,1]$, the map from $\mathcal{T}(A)$ to $\mathcal{S}(K_0(A))$ is not extreme-point preserving, so A cannot be an AH algebra. The tensor product of A with any UHF algebra has the same properties and has (SP).

I.3.7.16. Simple unital ASH algebras which are projectionless can also be constructed. The best example is the Jiang-Su algebra ().

I.4. Tensor Products, Nuclearity, and the UCT

The theory of tensor products of C*-algebras is fraught with a surprising number of technical complications, but the theory ends up in a rather satisfactory form, and behaves very nicely for a large and natural class of C*-algebras called "nuclear C*-algebras," which also perhaps surprisingly arise in several other aspects of the subject. Tensor products of C*-algebras were first studied by T. Turumaru [**?**]; in this and some other early references they were called "direct products."

We technically will not need any of the results of this section for the Classification Theorem, but much of the later work will not make too much sense without some understanding of this large body of preliminary material.

This section is partly excerpted from [Bla06], which contains more details. See also [BO08] for a comprehensive treatment.

I.4.1. Algebraic and Spatial Tensor Products

I.4.1.1. If A and B are C*-algebras, we can form their algebraic tensor product $A \odot B$ over $\mathbb{C}$. $A \odot B$ has a natural structure as a *-algebra with multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

and involution $(a \otimes b)^* = a^* \otimes b^*$. We want to show the existence of a C*-norm on $A \odot B$ and determine the extent to which it is unique. If γ is a C*-norm on $A \odot B$, we will write $A \otimes_\gamma B$ for the completion.

I.4.1.2. $A \odot B$ has the usual universal property for bilinear maps. As an algebra, it has the universal property that whenever π_A and π_B are *-homomorphisms from A and B respectively, to a complex *-algebra C, such that $\pi_A(A)$ and $\pi_B(B)$ commute, then there is a unique *-homomorphism π from $A \odot B$ to C such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$ for all $a \in A$, $b \in B$. Taking $C = \mathcal{B}(\mathcal{H})$, we get *-representations of $A \odot B$ and hence induced C*-seminorms.

The Minimal or Spatial Tensor Product

I.4.1.3. A standard way to generate such representations is via tensor products of Hilbert spaces: if π_A and π_B are representations of A and B on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, we can form the representation $\pi = \pi_A \otimes \pi_B$ of $A \odot B$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by $\pi(a \otimes b) = \pi_A(a) \otimes \pi_B(b)$. If π_A and π_B are faithful, then it is not difficult to show that $\pi_A \otimes \pi_B$ is faithful on $A \odot B$, so $A \odot B$ has at least one C*-norm. Also, for any π_A and π_B we have $\|(\pi_A \otimes \pi_B)\left(\sum_{i=1}^n a_i \otimes b_i\right)\| \leq \sum_{i=1}^n \|a_i\|\|b_i\|$, so the norm

$$\left\|\sum_{i=1}^n a_i \otimes b_i\right\|_{\min} = \sup \left\|(\pi_A \otimes \pi_B)\left(\sum_{i=1}^n a_i \otimes b_i\right)\right\|$$

(over all representations π_A of A and π_B of B) is finite and hence a C*-norm, called the *spatial* norm on $A \odot B$; it is also called the *minimal C*-norm* because it indeed turns out to be the smallest C*-norm on $A \odot B$. (A consequence of the minimality of $\|\cdot\|_{\min}$ is that if π_A and π_B are any faithful representations of A and B respectively, and $x \in A \odot B$, then $\|(\pi_A \otimes \pi_B)(x)\| = \|x\|_{\min}$, i.e. the spatial norm is independent of the faithful representations chosen.) The completion of $A \odot B$ with respect to this norm is written $A \otimes B$, or $A \otimes_{\min} B$ for emphasis, and called the *minimal* or *spatial* tensor product of A and B.

I.4.2. The Maximal Tensor Product

There is also a maximal C*-norm on $A \odot B$. Indeed, every *-representation of $A \odot B$ comes from a pair of commuting representations of A and B as above:

I.4.2.1. Theorem. Let A and B be C*-algebras, and π a nondegenerate representation of $A \odot B$ on a Hilbert space $\mathcal{H}$. Then there are unique nondegenerate representations π_A of A and π_B of B on $\mathcal{H}$ such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$ for all a, b. If π is a factor representation, then π_A and π_B are also factor representations. Thus every C*-seminorm γ on $A \odot B$ extends uniquely to a C*-seminorm on $A^\dagger \odot B^\dagger$, and satisfies $\gamma(a \otimes b) \leq \|a\|\|b\|$ for all $a \in A$, $b \in B$.

Thus most questions about tensor products can be reduced to the unital case.

I.4.2.2. So $\|\pi\left(\sum_{i=1}^n a_i \otimes b_i\right)\| \leq \sum_{i=1}^n \|a_i\|\|b_i\|$ for any representation π of $A \odot B$, and hence

$$\left\|\sum_{i=1}^n a_i \otimes b_i\right\|_{\max} = \sup \left\|\pi\left(\sum_{i=1}^n a_i \otimes b_i\right)\right\|$$

where the supremum is taken over all representations, is a (finite) C*-norm on $A \odot B$ which is the largest possible C*-norm. The completion is denoted $A \otimes_{\max} B$, and called the *maximal tensor product* of A and B. If γ is any C*-norm on $A \odot B$, there is a natural quotient map from $A \otimes_{\max} B$ to $A \otimes_\gamma B$ which is the identity map on $A \odot B$; in particular, there is a natural quotient map from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ (in fact, if γ is any C*-norm on $A \odot B$, there are quotient maps $A \otimes_{\max} B \to A \otimes_\gamma B \to A \otimes_{\min} B$.)

I.4.2.3. It is obvious that $A \otimes_{\max} B$ and $A \otimes_{\min} B$ are unital if (and only if) both A and B are unital. In this case, there are natural embeddings $A \to A \otimes 1_B \subseteq A \otimes_\gamma B$ and $B \to 1_A \otimes B \subseteq A \otimes_\gamma B$ for any C*-norm γ on $A \odot B$. If A and B are unital C*-algebras, then $A \otimes_{\max} B$ can be described as the universal unital C*-algebra generated by unital copies of A and B which commute; it is harder to describe $A \otimes_{\max} B$ as a universal C*-algebra in the nonunital case since A and B do not naturally embed.

I.4.3. States on Tensor Products

I.4.3.1. $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ can be alternately described using appropriate positive linear functionals on $A \odot B$, where a positive linear functional on $A \odot B$ is a linear functional f such that $f(x^*x) \geq 0$ for all $x \in A \odot B$. The set of positive linear functionals on $A \odot B$ is denoted $(A \odot B)^*_+$. The set $(A \odot B)^*_+$ is a cone in the algebraic dual $(A \odot B)^d$ which is closed in the topology of pointwise convergence (weak^d topology).

If A and B are unital, a *state* on $A \odot B$ is an element $\phi \in (A \odot B)^*_+$ with $\phi(1 \otimes 1) = 1$. The set of states, denoted $\mathcal{S}(A \odot B)$, is a weak^d-compact convex set.

$\mathcal{S}(A \odot B)$ can be identified with $\mathcal{S}(A \otimes_{\max} B)$ (there are some technicalities to check); the identification is an affine homeomorphism for the weak^d and weak-* topologies.

I.4.3.2. If ϕ and ψ are positive linear functionals on A and B respectively, then the functional $\phi \otimes \psi$ defined by

$$(\phi \otimes \psi)\left(\sum_i a_i \otimes b_i\right) = \sum_i \phi(a_i)\psi(b_i)$$

is well defined and positive (this follows easily by considering the representation $\pi_\phi \otimes \pi_\psi$ and the vector $\xi_\phi \otimes \xi_\psi$). Let $(A \odot B)^*_{++}$ be the subset of $(A \odot B)^*_+$ of pointwise limits of convex combinations of such product functionals (one can just use multiples of products of pure states). Then for $x \in A \odot B$ we have:

$$\|x\|^2_{\max} = \sup\left\{\frac{\phi(y^*x^*xy)}{\phi(y^*y)} : \phi \in (A \odot B)^*_+, y \in A \odot B, \phi(y^*y) \neq 0\right\}$$

$$\|x\|^2_{\min} = \sup\left\{\frac{\phi(y^*x^*xy)}{\phi(y^*y)} : \phi \in (A \odot B)^*_{++}, y \in A \odot B, \phi(y^*y) \neq 0\right\}$$

$$= \sup\left\{\frac{(\phi \otimes \psi)(y^*x^*xy)}{(\phi \otimes \psi)(y^*y)} : \phi \in \mathcal{P}(A), \psi \in \mathcal{P}(B), y \in A \odot B, (\phi \otimes \psi)(y^*y) \neq 0\right\}.$$

I.4.3.3. So $\mathcal{S}(A \otimes_{\min} B)$ [resp. $(A \odot B)^*_{++}$] can be identified with the closure of $\mathcal{S}(A \odot B) \cap (A^* \odot B^*)$ in $\mathcal{S}(A \odot B)$ [resp. the closure of $(A \odot B)^*_+ \cap (A^* \odot B^*)$ in $(A \odot B)^*_+$.] More generally, if γ is any cross norm on $A \odot B$, then $\mathcal{S}(A \otimes_\gamma B)$ can be identified with the set of states on $A \odot B$ which are continuous for γ. This subset completely determines γ by the Hahn-Banach Theorem.

I.4.3.4. There is a standard identification $f \leftrightarrow T_f$ of $(A \odot B)^d$ with the set $\mathrm{Hom}(A, B^d)$ of linear maps from A to B^d, where $[T_f(a)](b) = f(a \otimes b)$. Recall that B^* has a structure as a matrix ordered space ().

I.4.3.5. Proposition.

(i) Let $f \in (A \odot B)^d$. Then $f \in (A \odot B)^*_+$ if and only if T_f is a completely positive map from A to $B^* \subseteq B^d$.

(ii) The map $f \to T_f$ is an affine homeomorphism from $(A \odot B)^*_+$ with the weak^d-topology onto $CP(A, B^*)$ with the topology of pointwise weak-* convergence.

I.4.4. Nuclear C*-Algebras

I.4.4.1. Definition. A C*-algebra A is *nuclear* if, for every C*-algebra B, there is a unique C*-norm on $A \odot B$ (i.e. the quotient map from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ is an isomorphism).

If A is nuclear, the notation $A \otimes B$ is unambiguous for any B.

I.4.4.2. The term "nuclear" is in analogy with the notion of a nuclear space in Grothendieck's theory of tensor products of topological vector spaces [**?**], although an infinite-dimensional C*-algebra is never a nuclear space in the sense of Grothendieck.

I.4.4.3. EXAMPLES. Let A be any C*-algebra. Then $\mathbb{M}_n \odot A$ can be identified with $M_n(A)$ in the standard way: if $\{e_{ij}\}$ are the standard matrix units of $\mathbb{M}_n$, then

$$\left(\sum_{i,j=1}^{n} e_{ij} \otimes a_{ij}\right) \mapsto (a_{ij})$$

is an isomorphism. Since $M_n(A)$ is a C*-algebra under the norm of I.1.2.3., this norm is the unique C*-norm on $\mathbb{M}_n \odot A$, and in particular agrees with $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$, and

$$\mathbb{M}_n \otimes_{\max} A = \mathbb{M}_n \otimes_{\min} A = \mathbb{M}_n \odot A$$

(written $\mathbb{M}_n \otimes A$). So $\mathbb{M}_n$ is nuclear. Similarly, it is easily seen that $A \otimes \mathbb{K}$ in the sense of I.1.7.12. agrees with $A \otimes_{\max} \mathbb{K}$ and $A \otimes_{\min} \mathbb{K}$ in the tensor product sense (although it is not the same as $A \odot \mathbb{K}$ in general), so $\mathbb{K}$ is nuclear.

I.4.4.4. There is a partial converse to this example. Let $\{e_{ij} : 1 \leq i, j \leq n\}$ be a set of matrix units of type $\mathbb{M}_n$ in a C*-algebra A. Then the e_{ij} generate a (possibly nonunital) C*-subalgebra isomorphic to $\mathbb{M}_n$. If A is unital and $\sum_i e_{ii} = 1$, then for $a \in A$ set $a_{ij} = e_{1i}ae_{j1} \in e_{11}Ae_{11}$; then $a \mapsto (a_{ij})$ is an isomorphism from A onto $M_n(e_{11}Ae_{11}) \cong \mathbb{M}_n \otimes e_{11}Ae_{11}$.

Another important example is:

I.4.4.5. THEOREM. [Tak64] Let $A = C_0(X)$ be commutative. Then A is nuclear, and for any C*-algebra B, $A \otimes B$ can be identified with $C_0(X, B)$ (I.1.2.3.) under the map $(f \otimes b)(x) = f(x)b$. In particular, if $B = C_0(Y)$ is also commutative, then $C_0(X) \otimes C_0(Y) \cong C_0(X \times Y)$ under the identification

$$(f \otimes g)(x, y) = f(x)g(y) \ .$$

To show A is nuclear, we may assume for simplicity that A and B are unital (I.4.2.1.). Let ϕ be any pure state of $A \otimes_{\max} B$, and π_A and π_B the restrictions to A and B (I.4.2.1.) of the irreducible GNS representation π_ϕ. Since A is commutative, it follows from I.4.2.1. that $\pi_A(A) \subseteq \mathbb{C}I$, and so $\pi_\phi(a \otimes b) = \chi(a)\pi_B(b)$ for all $a \in A$, $b \in B$, where $\chi \in \mathcal{P}(A) = \hat{A}$. Therefore $\phi(a \otimes b) = \chi(a)\psi(b)$ for some $\psi \in \mathcal{P}(B)$ ($\psi(b) = \langle \pi_\phi(1 \otimes b)\xi_\phi, \xi_\phi \rangle$), i.e. $\phi = \chi \otimes \psi$. Thus $\mathcal{P}(A \otimes_{\max} B) \cong X \times \mathcal{P}(B)$, and every pure state of $A \otimes_{\max} B$ factors through $A \otimes_{\min} B$, so $\|\cdot\|_{\max} = \|\cdot\|_{\min}$ on $A \odot B$. We also obtain that $\mathrm{Prim}(A \otimes_{\max} B) \cong X \times \mathrm{Prim}(B)$, and it is routine to verify the isomorphism with $C_0(X, B)$. Now let γ be a C*-norm and π the quotient map from $A \otimes_{\max} B$ onto $A \otimes_\gamma B$. π corresponds to a closed set in $\mathrm{Prim}(A \otimes_{\max} B)$; if this subset is proper, since the topology is the product topology there are ideals I in A and J in B such that $\pi|_{I \odot J} = 0$. But if a and b are nonzero elements of I and J respectively, $\gamma(a \otimes b) \neq 0$ since γ is a norm, a contradiction.

I.4.4.6. The class of nuclear C*-algebras is closed under most of the standard operations: extensions and tensor products (elementary observations), quotients (a deep theorem, cf. I.4.4.13.), hereditary C*-subalgebras (in particular, closed ideals), inductive limits (the case of injective connecting maps is elementary, but the general case is rather delicate (cf. I.4.4.13.)), and crossed products by amenable groups and connected groups. As a corollary, since $\mathbb{K}$ is nuclear, the class of nuclear C*-algebras is closed under stable isomorphism. It is thus not so easy to give examples of nonnuclear C*-algebras. The most elementary examples are $C^*(\mathbb{F}_n)$ and $C_r^*(\mathbb{F}_n)$ ($n > 1$). It is also true that $\mathcal{B}(\mathcal{H})$ is nonnuclear if $\mathcal{H}$ is infinite-dimensional. In fact, "most" C*-algebras are nonnuclear, as is clear from the characterizations of I.4.4.12.. A C*-subalgebra of a nuclear C*-algebra is not nuclear in general; in fact, every C*-algebra which is not Type I contains a nonnuclear C*-subalgebra [Bla85a] (Type I C*-algebras are nuclear, and a C*-subalgebra of a Type I C*-algebra is Type I, hence nuclear).

I.4.4.7. A is nuclear if and only if $(A \odot B)^*_{++} = (A \odot B)^*_+$ for every B. This has an alternate description which motivates the connection between nuclearity and completely positive approximations:

I.4.4.8. Proposition. Let π and ρ be representations of C*-algebras A and B on $\mathcal{H}$ and $\mathcal{H}'$ respectively, giving a representation $\pi \otimes \rho$ of $A \odot B$ (which extends to $A \otimes_{\min} B$) on $\mathcal{H} \otimes \mathcal{H}'$. Let ξ be a unit vector in $\mathcal{H} \odot \mathcal{H}'$, and ϕ the corresponding vector state on $A \odot B$. Then the map $T_\phi : A \to B^*$ of I.4.3.4. is a completely positive finite-rank contraction. Every completely positive finite-rank contraction from A to B^* arises in this manner.

I.4.4.9. Corollary. Let A and B be C*-algebras. Identify $(A \odot B)^*_+$ with $CP(A, B^*)$ as in I.4.3.4.. Then $(A \odot B)^*_{++}$ is the closure in $CP(A, B^*)$ (with the topology of pointwise weak-* convergence) of the finite-rank completely positive maps. In particular, A is nuclear if and only if the finite-rank maps are dense in $CP(A, B^*)$ for every B.

Thus nuclearity is a certain kind of order-theoretic approximate finite-dimensionality. In fact, nuclearity can be characterized by certain identity maps being approximable by completely positive finite-rank contractions; see I.4.4.17..

Here are some extensions of the notion of nuclearity:

I.4.4.10. Definition.

(i) A C*-algebra A is *quasinuclear* if, whenever $B \subseteq B_1$ are C*-algebras, the "inclusion" map $A \otimes_{\max} B \to A \otimes_{\max} B_1$ is isometric.

(ii) A C*-algebra A is *seminuclear* if, whenever $A \subseteq A_1$ and B are C*-algebras, the "inclusion" map $A \otimes_{\max} B \to A_1 \otimes_{\max} B$ is isometric.

Any nuclear C*-algebra is quasinuclear and seminuclear. It is clear that a C*-subalgebra of a nuclear C*-algebra is quasinuclear or seminuclear if and only if it is nuclear.

Actually, it is a (deep) fact that any quasinuclear C*-algebra is nuclear (I.4.4.12.). But there are seminuclear C*-algebras which are not nuclear: for example, any injective C*-algebra is seminuclear.

Nuclearity and the Second Dual

I.4.4.11. If A is a C*-algebra, the second dual A^{**} of A has a natural multiplication making it into a W*-algebra (). It is isomorphic to $\pi(A)''$, where π is the *universal representation* of A, the direct sum of all GNS representations from all states of A. If A ($\neq \mathbb{C}$) is separable, then every (nondegenerate) representation ρ of A on a separable Hilbert space is equivalent to a subrepresentation of the universal representation, and thus $\rho(A)''$ is isomorphic to a direct summand of A^{**}. (But the universal representation acts on a nonseparable Hilbert space unless $A = \mathbb{C}$, and A^{**} itself rarely has separable predual.)

The following crucial consequence of the deep result I.2.3.22. for C*-algebras was obtained by Choi and Effros [**?**] (some additional argument was needed for the nonseparable case):

I.4.4.12. Theorem. Let A be a C*-algebra. The following are equivalent:

(i) A is nuclear.

(ii) A is quasinuclear (I.4.4.10.).

(iii) A^{**} is semidiscrete (**??**).

(iv) A^{**} is injective.

I.4.4.13. Corollary. A quotient of a nuclear C*-algebra is nuclear.

Proof: If J is a closed ideal in a C*-algebra A, then

$$A^{**} \cong J^{**} \oplus (A/J)^{**}$$

(**??**). If A is nuclear, then A^{**} is injective, so $(A/J)^{**}$ is also injective.

Factorization Through Matrix Algebras

Nuclearity turns out to be characterized by the existence of completely positive finite-rank approximations of the identity (cf. I.4.4.9.).

I.4.4.14. Definition. If A and B are C*-algebras, a completely positive contraction $\phi : A \to B$ is *nuclear* if ϕ approximately factors through matrix algebras in the point-norm topology, i.e. if, for any $x_1, \dots, x_k \in A$ and $\epsilon > 0$, there is an n and completely positive contractions

$$\alpha : A \to \mathbb{M}_n,\ \beta : \mathbb{M}_n \to B$$

such that $\|\phi(x_j) - \beta \circ \alpha(x_j)\| < \epsilon$ for $1 \leq j \leq k$.

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\alpha"'] & & B \\ & \mathbb{M}_n \arrow[ur, "\beta"'] & \end{tikzcd}$$

I.4.4.15. Definition. A C*-algebra A has the *completely positive approximation property* or *CPAP* if the identity map on A is nuclear, i.e. if, for any $x_1, \dots, x_k \in A$ and $\epsilon > 0$, there is an n and completely positive contractions

$$\alpha : A \to \mathbb{M}_n, \ \beta : \mathbb{M}_n \to A$$

such that $\|x_j - \beta \circ \alpha(x_j)\| < \epsilon$ for $1 \le j \le k$.

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & A \\ & \mathbb{M}_n \arrow[ur, "\beta"'] & \end{tikzcd}$$

I.4.4.16. If F is a finite-dimensional C*-algebra, then F can be embedded as a C*-subalgebra of $\mathbb{M}_n$ for some n, and there is a conditional expectation (I.2.3.11.) from $\mathbb{M}_n$ onto F; thus in the definitions I.4.4.14. and I.4.4.15. the $\mathbb{M}_n$ can be equivalently replaced by a general finite-dimensional C*-algebra.

The Main Theorem

The principal theorem of the theory of nuclear C*-algebras is a combination of several deep results:

I.4.4.17. Theorem. Let A be a C*-algebra. The following are equivalent:

(i) A is nuclear.

(ii) The identity map on A is a point-norm limit of completely positive finite-rank contractions.

(iii) A has the CPAP.

(iv) A^{**} is injective.

(v) A is C*-amenable (I.4.5.27.).

I.4.4.17. shows that the nuclear C*-algebras form an extremely natural and well-behaved class; further evidence comes from the results in **??** about group C*-algebras and crossed products. Nuclear C*-algebras also have additional nice structure properties; perhaps the most important of these is the Choi-Effros Lifting Theorem (I.4.4.21.).

I.4.4.18. From the CPAP philosophy that nuclear C*-algebras are "approximately finite-dimensional" in an order-theoretic sense, it might be thought that classification of nuclear C*-algebras is just one step up from classification of AF algebras, and can be done in roughly the same way. But this philosophy is at least somewhat misleading (it is up in the air just how misleading). For one thing, there is a huge difference between the order-theoretic "approximate finite-dimensionality" characterizing nuclearity and the algebraic approximate finite-dimensionality of AF algebras, illustrated by the existence of purely infinite nuclear C*-algebras (not to mention the UCT problem). In the stably finite case, the analogy may be closer, as discussed in I.7.3.5.. But even a classification in the stably finite case would include a classification of commutative C*-algebras, essentially the opposite extreme to the simple AF algebras; commutative C*-algebras are classified by the trace space with no K-theory needed. It is thus remarkable that the AF and commutative classifications can be extended to a large class of simple nuclear C*-algebras, although not all separable simple nuclear C*-algebras (it is still unclear how large a class is left unclassified), by combining the trace space with K-theory. Some efforts have been made to begin filling in the gap between the commutative and simple cases, e.g. by classifying certain C*-algebras whose primitive ideal space is a finite space of small cardinality (), but even in the case of only finitely many ideals the general picture is not yet even on the horizon.

Completely Positive Liftings

One of the most useful consequences of nuclearity is the Choi-Effros lifting theorem for completely positive maps into quotients. This result has important applications in K-theory.

I.4.4.19. Let A and B be C*-algebras, and J a closed ideal of B. A completely positive contraction (*cp*-contraction) $\phi : A \to B/J$ is *liftable* (to B) if there is a *cp*-contraction $\psi : A \to B$ such that $\phi = \pi \circ \psi$, where $\pi : B \to B/J$ is the quotient map. ψ is called a *lifting* of ϕ.

A has the *lifting property* if ϕ is liftable for every B, J, and ϕ.

I.4.4.20. An important special case comes when $A = B/J$ and ϕ is the identity map. A lifting ψ is then a completely positive cross section for π, so the extension $0 \to J \to B \to B/J \to 0$ is *semisplit*. Semisplit extensions are well behaved for many purposes related to K-theory.

I.4.4.21. Theorem.[Choi-Effros Lifting] Let A and B be C*-algebras, and J a closed ideal of B. If A is separable and $\phi : A \to B/J$ is a nuclear completely positive contraction, then ϕ is liftable to B. In particular, every separable nuclear C*-algebra has the lifting property.

I.4.4.22. Corollary. If $0 \to J \to B \to B/J \to 0$ is an extension of C*-algebras, and B/J is separable and nuclear, then the extension is semisplit.

Using I.4.4.13., it suffices that B is nuclear (and B/J separable).

Exact C*-Algebras

I.4.4.23. If A and B are algebras over $\mathbb{C}$ (or over any field), and J is an ideal of B, it is an elementary fact about tensor products that the natural map from $A \odot J$ into $A \odot B$ is injective, and that $(A \odot B)/(A \odot J)$ is naturally identified with $A \odot (B/J)$ as an algebra. In other words, the sequence

$$0 \longrightarrow A \odot J \longrightarrow A \odot B \longrightarrow A \odot (B/J) \longrightarrow 0$$

is exact, i.e. algebraic tensor product by A is an *exact functor*. [More generally, algebraic tensor product is exact for vector spaces over a field, but not in general for modules over a ring, even $\mathbb{Z}$ (the derived functor Tor measures the deviation from exactness).]

I.4.4.24. It is also an immediate consequence of the definition of the minimal C*-tensor product that if A and B are C*-algebras and J is a closed ideal of B, then the closure of $A \odot J$ in $A \otimes_{\min} B$ is isometrically isomorphic to $A \otimes_{\min} J$. Thus $A \odot (B/J)$ embeds as a dense subalgebra of $(A \otimes_{\min} B)/(A \otimes_{\min} J)$, and the quotient norm defines a C*-cross norm α on $A \odot (B/J)$; so the exact sequence of I.4.4.23. completes to an exact sequence

$$0 \longrightarrow A \otimes_{\min} J \longrightarrow A \otimes_{\min} B \longrightarrow A \otimes_{\alpha} (B/J) \longrightarrow 0$$

for some C*-cross norm α on $A \odot (B/J)$. However, α is not the minimal cross norm in general: it was shown in [?] that if $A = B = C^*(\mathbb{F}_2)$ and J is the kernel of the quotient map from B to $C_r^*(\mathbb{F}_2)$, the cross norm α is not minimal. Since $A \otimes_{\min} (B/J)$ is a quotient of $A \otimes_{\alpha} (B/J)$ whatever α may be, there is always a well-defined sequence

$$0 \longrightarrow A \otimes_{\min} J \longrightarrow A \otimes_{\min} B \longrightarrow A \otimes_{\min} (B/J) \longrightarrow 0$$

but it is *not* exact in the middle in general.

I.4.4.25. Definition. A C*-algebra A is *exact* if, for every C*-algebra B and closed ideal J of B, the sequence

$$0 \longrightarrow A \otimes_{\min} J \longrightarrow A \otimes_{\min} B \longrightarrow A \otimes_{\min} (B/J) \longrightarrow 0$$

is exact (in other words, $\alpha = \min$ for every B and J), i.e. minimal tensor product with A is an exact functor.

I.4.4.26. If A is nuclear, then min is the only C*-cross norm on $A \odot (B/J)$ for any B and J, so A is exact. The converse is false: it is known () that $C_r^*(\mathbb{F}_2)$ is exact but not nuclear.

More generally, since if A is a C*-subalgebra of D and B is any C*-algebra, the closure of $A \odot B$ in $D \otimes_{\min} B$ is $A \otimes_{\min} B$, it follows easily that a C*-subalgebra of an exact C*-algebra is exact. Thus every subnuclear C*-algebra (C*-subalgebra of a nuclear C*-algebra) is exact. Conversely, a deep theorem of Kirchberg () states that every separable exact C*-algebra embeds in the Cuntz algebra O_2, which is nuclear, so every separable exact C*-algebra is subnuclear.

I.4.5. Amenable C*-Algebras

A property closely related to nuclearity is amenability:

I.4.5.27. Definition. A C*-algebra A is *amenable* (C*-amenable) if every (bounded) derivation from A to a dual Banach A-module is inner.

Fundamental work on amenability for Banach algebras was done by B. Johnson, individually [?] and in collaboration with R. Kadison and J. Ringrose [?], who had begun the cohomology theory of Banach algebras [?], [?]; cf. [Bla06].

I.4.5.28. C*-Amenability is also characterized by the existence of a "virtual diagonal" [Bla06, IV.3.3.2].

I.4.5.29. Theorem. A C*-algebra is amenable if and only if it is nuclear.

Both directions of the equivalence of amenability and nuclearity are nontrivial and require some heavy machinery. Connes, using [?], proved that amenable implies nuclear in 1978 [Con78]. The converse proved elusive, and was finally proved by Haagerup in 1983 [Haa83]. There is a well-known fallacious argument for this direction (cf. [?], [Bla06, IV.3.3.14]).

One indication that the equivalence is not easy is that it is virtually trivial that a quotient of an amenable C*-algebra is amenable, while it is (apparently) a deep theorem that a quotient of a nuclear C*-algebra is nuclear.

I.4.5.30. The C*-algebra analog of I.2.3.21. is false. There is a refined notion of C*-amenability, called *strong amenability*. It is also virtually trivial that a quotient of a strongly amenable C*-algebra is strongly amenable. If G is an amenable discrete group, then $C^*(G)$ is strongly amenable; so if A is a unital C*-algebra generated by a group G which is amenable in the discrete topology, then A is a quotient of $C^*(G)$ and is thus strongly amenable. But a strongly amenable unital C*-algebra has a trace; thus Kirchberg algebras are amenable but not strongly amenable, and cannot be generated by a discrete amenable group of unitaries. The unitary group of a unital nuclear C*-algebra is amenable in the weak topology, and conversely [?].

Terminology

I.4.5.31. In light of the equivalence of nuclearity and amenability, there is a question which name should be used generically for this class of C*-algebras. The name *nuclear* is traditional. Many authors in the classification program use the term *amenable* instead. It is strictly a matter of taste which to use. I am personally in the habit (and I admit it is merely a *habit*) of using *nuclear*, and I will in my book, but I take no issue with authors who prefer the other convention, for which there are attractive arguments (especially that they are the C*-algebras "amenable" to classification!).

However, I do have a problem with *defining* amenable C*-algebras to be ones satisfying the CPAP (), as is done in some references. If the term *amenable* is used, it should at least be given the proper definition, and the (quite nontrivial) equivalence with the CPAP mentioned explicitly. (It is technically equally incorrect to define nuclearity by the CPAP, although this condition is more closely related to the proper definition and the equivalence, while nontrivial, is not as deep. The CPAP is really the crucial property needed in the classification program.)

I think that besides force of habit, there are two arguments in favor of using *nuclear* generically instead of *amenable*:

(i) The term *amenable* is used in more than one sense in the subject of operator algebras (e.g. I.2.3.20.); this usage should properly be called *C*-amenable*. There is no such ambiguity with *nuclear*.

(ii) The notion of *finite nuclear dimension* is a critical part of the classification theorem, and meshes more nicely with *nuclear*.

End of sermon.

I.4.6. The Bootstrap Class and the UCT

The definition and various characterizations of nuclearity for C*-algebras are structural rather than constructive. Perhaps the most outstanding open question in the subject of operator algebras is whether there is a constructive characterization of nuclearity. This is a version of a question known as the *UCT problem*.

I.4.6.1. It is either elementary or a standard result that

(i) All finite-dimensional C*-algebras are nuclear.

(ii) All commutative C*-algebras are nuclear [Tak64].

(iii) The class of nuclear C*-algebras is closed under

closed ideals, quotients, and extensions.
(finite) tensor products.
inductive limits (hence infinite tensor products).
stable isomorphism.
crossed products by amenable groups ().

Some of these results follow easily from one or another of the characterizations of nuclearity.

I.4.6.2. Closure under ideals, quotients, and extensions has an interesting historical twist. It is nearly trivial that a (closed) ideal in a nuclear C*-algebra is nuclear, since it is easily verified that if J is a closed ideal in A, and B is any C*-algebra, then the closure of $J \odot B$ in $A \otimes_{\max} B$ and in $A \otimes_{\min} B$ is isomorphic to $J \otimes_{\max} B$ and $J \otimes_{\min} B$ respectively. But quotients and extensions are not so obvious. The fact that quotients and extensions of nuclear C*-algebras are nuclear was first published in [CE76b] (cf. [CE77b]), as a corollary of I.4.4.12. which depends on the very deep result I.2.3.22.. There is no more elementary proof known for quotients; however, there is a simple elementary proof for extensions using the Five Lemma [Bla06, IV.3.1.3]. It is not true in general that a C*-subalgebra of a nuclear C*-algebra is nuclear; in fact, every C*-algebra which is not Type I has a nonnuclear C*-subalgebra [Bla85a]. (The C*-subalgebras of nuclear C*-algebras are precisely the exact C*-algebras (I.4.4.25.).)

I.4.6.3. As a result of these facts, every C*-algebra built up from finite-dimensional or commutative C*-algebras by standard constructions is nuclear. In particular, all AH and ASH algebras are nuclear.

I.4.6.4. Definition. Let $\mathcal{B}$ be the smallest class of (separable) C*-algebras such that

(i) All finite-dimensional C*-algebras are in $\mathcal{B}$.

(ii) All (separable) commutative C*-algebras are in $\mathcal{B}$.

(iii) $\mathcal{B}$ is closed under

(a) the *two-out-of-three property*: if

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is an exact sequence of C*-algebras, and two of J, A, A/J are in $\mathcal{B}$, then so is the third.

(b) (finite) tensor products.

(c) inductive limits (hence infinite tensor products).

(d) stable isomorphism.

(e) crossed products by $\mathbb{Z}$.

$\mathcal{B}$ is called the *Small Bootstrap Class*. (There is some redundancy in this definition, e.g. (i) follows from (ii) and closure under stable isomorphism and extensions. Actually (iiib) is also redundant, cf. [Bla98, 22.3.5(f)].)

I.4.6.5. One might find it curious that $\mathcal{B}$ is only assumed closed under the two-out-of-three property instead of being closed under ideals, quotients, and extensions, and only assumed closed under crossed products by $\mathbb{Z}$, not arbitrary amenable groups. An explanation is in (). There are many possible variations in the definition, such as replacing (ii) by closure under homotopy equivalence. (Closure under homotopy equivalence and quotients would give all C*-algebras, since if A is any C*-algebra, the cone $CA = C_0((0,1],A)$ is homotopy equivalent to the zero C*-algebra, which is in $\mathcal{B}$, and A is a quotient of CA. But closure under homotopy equivalence and the two-out-of-three property would at least give $\mathcal{B}$ but does not give an obviously larger class. See [Bla98, 22.3.5].)

I.4.6.6. **Small Bootstrap Problem:** Is $\mathcal{B}$ the class of all separable nuclear C*-algebras?

There is no separable nuclear C*-algebra known not to be in $\mathcal{B}$.

I.4.6.7. There is an analogy, widely believed to be imperfect, with amenable groups. Amenability for a (discrete) group is a structural property. The class of amenable groups is closed under standard operations analogous to the ones of I.4.6.4.; the smallest class of groups containing $\mathbb{Z}$ and closed under these operations is called the class of elementary amenable groups. It was an important question whether every amenable group is elementary amenable, solved in the negative in ().

The amenable group case is superficially closely related to the nuclear C*-algebra case, e.g. by the fact that $C^*(G)$ is nuclear if and only if G is amenable (). However, it is known that if G is a countable amenable group, then $C^*(G) \in \mathcal{B}$, even if G is not elementary amenable [Tu99]. So at least in this sense the analogy is imperfect. My impression is that operator algebra specialists regard the existence of amenable but not elementary amenable groups as having no bearing or relevance to the Small Bootstrap Problem.

KK-Theory

The Large Bootstrap Class

Equipped with KK-theory, we can define a potentially larger bootstrap class:

I.4.6.8. Definition. Let $\mathcal{N}$ be the smallest class of separable nuclear C*-algebras such that

(i) $\mathbb{C} \in \mathcal{N}$.

(ii) $\mathcal{N}$ has the two-out-of-three property.

(iii) $\mathcal{N}$ is closed under KK-equivalence.

$\mathcal{N}$ is called the *Large Bootstrap Class*, or *UCT Class*.

I.4.6.9. It can be shown that $\mathcal{B} \subseteq \mathcal{N}$ [Bla98, 22.3.5], and that $\mathcal{N}$ is closed under homotopy equivalence and stable isomorphism (since these are stronger than KK-equivalence), and under tensor products, inductive limits, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$.

I.4.6.10. We may also define an even larger bootstrap class, the smallest class $\mathcal{H}$ of separable (not necessarily nuclear) C*-algebras closed under (i)–(iii). $\mathcal{H}$ contains many nonnuclear C*-algebras, and could potentially even contain more nuclear C*-algebras than $\mathcal{N}$.

It is known that there are separable C*-algebras which are not in $\mathcal{H}$. But no separable nuclear C*-algebra is known not to be in $\mathcal{N}$ (or even $\mathcal{B}$), and at present we have no idea how such a C*-algebra (if there is one) could be constructed, since it could not be by known methods (although see ()).

I.4.6.11. **UCT Problem:** Is every separable nuclear C*-algebra in $\mathcal{N}$?

Of course, a positive answer to the Small Bootstrap Problem I.4.6.6. would give a positive answer to the UCT problem. But the UCT problem is potentially much weaker, and may stand a much better chance of having a (provably) positive answer.

I.4.6.12. The reason the Small Bootstrap Problem is important for classification is obvious: all known separable simple nuclear C*-algebras are in $\mathcal{B}$, and exhaust the Elliott invariant, so if there is a separable simple nuclear C*-algebra not in $\mathcal{B}$, it could not be classifiable. The reason the UCT Problem is at least as important, and the reason why it is called the UCT problem, is that theorems from KK-theory can be applied to show that simple C*-algebras in $\mathcal{N}$ with finite nuclear dimension are classifiable, and in particular in $\mathcal{B}$ (suggesting that all simple C*-algebras in $\mathcal{N}$ are in $\mathcal{B}$, and maybe that $\mathcal{N} = \mathcal{B}$). One of the theorems of KK-theory is called the Universal Coefficient Theorem and gives the UCT Problem its name.

Universal Coefficient Theorems and Künneth Theorems

There are some theorems about K-theory and KK-theory which can be proved by bootstrap methods and which thus hold for C*-algebras in $\mathcal{N}$. These theorems were proved by Rosenberg and Schochet (special cases were previously obtained by Cuntz and L. Brown). We simply state the theorems here; a full treatment can be found in [Bla98].

I.4.6.13. We will regard K-theory and KK-theory as $\mathbb{Z}_2$-graded theories to simplify notation: if A and B are (separable) C*-algebras, write

$$K_*(A) = K_0(A) \oplus K_1(A)$$

$$KK^*(A,B) = KK(A,B) \oplus KK^1(A,B)$$

as $\mathbb{Z}_2$-graded groups. If $G = G_0 \oplus G_1$ and $H = H_0 \oplus H_1$ are $\mathbb{Z}_2$-graded groups, there is a natural grading of $G \otimes_{\mathbb{Z}} H$ as

$$[(G_0 \otimes_{\mathbb{Z}} H_0) \oplus (G_1 \otimes_{\mathbb{Z}} H_1)] \oplus [(G_1 \otimes_{\mathbb{Z}} H_0) \oplus (G_0 \otimes_{\mathbb{Z}} H_1)]$$

and a natural grading of $\mathrm{Hom}(G,H)$ by grading-preserving/grading-reversing homomorphisms. There are natural degree zero homomorphisms

$$\alpha : K_*(A) \otimes_{\mathbb{Z}} K_*(B) \to K_*(A \otimes B)$$

$$\gamma : KK^*(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$$

(γ is defined by the Kasparov product $KK(\mathbb{C},A) \times KK(A,B) \to KK(\mathbb{C},B)$, and α has a straightforward K-theoretic definition).

I.4.6.14. One might hope that α and γ would be isomorphisms, but they cannot be in general for (essentially) homological algebra reasons. If the sequences are modified in the appropriate way to incorporate the homological algebra obstructions, sequences are obtained which are valid at least for C*-algebras in $\mathcal{N}$.

The additional ingredient is easiest to describe in the case of γ. If $\gamma(\tau) = 0$ for an extension τ, then the six-term K-theory sequence degenerates into two short exact sequences of the form

$$0 \longrightarrow K_i(B) \longrightarrow K_i(D) \longrightarrow K_i(A) \longrightarrow 0$$

and thus determines an element $\kappa(\tau) \in \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$ [this is the $\mathrm{Ext}^1_{\mathbb{Z}}$-group of homological algebra.] Note that κ reverses degree. The maps γ and κ are generalizations of the Adams d and e operations in topological K-theory.

The obstruction for α is an element of $\mathrm{Tor}^{\mathbb{Z}}_1$, the derived functor of the tensor product functor. It is natural to expect a $\mathrm{Tor}^{\mathbb{Z}}_1$ obstruction, since $\mathrm{Tor}^{\mathbb{Z}}_1$ measures the deviation from exactness of the tensor product functor on groups.

I.4.6.15. Theorem. [Universal Coefficient Theorem (UCT)] [RS87] Let A and B be separable C*-algebras, with $A \in \mathcal{N}$. Then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

The map γ has degree 0 and δ has degree 1 (it is the inverse of κ). The sequence is natural in each variable, and splits unnaturally. So if $K_*(A)$ is free or $K_*(B)$ is divisible, then γ is an isomorphism.

This result is called the "Universal Coefficient Theorem" since K-theory is an extraordinary cohomology theory and the theorem can be viewed as an analog of the usual Universal Coefficient Theorem of cohomology ().

I.4.6.16. Informally, the UCT can be interpreted as saying that for C*-algebras in $\mathcal{N}$, KK-theory is completely determined by K-theory. In fact, the following corollaries can be deduced:

I.4.6.17. Corollary. Let A and B be C*-algebras in $\mathcal{N}$. Then $A \sim_{KK} B$ if and only if $K_*(A) \cong K_*(B)$. In fact, any element of $KK(A, B)$ inducing an isomorphism on K-theory is a KK-equivalence.

Note that the isomorphism of $K_*(A)$ and $K_*(B)$ is only at the (graded) group level; no order on K_0 is included. Thus, for example, two AF algebras are KK-equivalent if and only if their dimension groups are isomorphic as groups, ignoring the order and scale completely.

Since separable commutative C*-algebras exhaust all possibilities for K-groups, we also obtain:

I.4.6.18. Corollary. Let A be a separable nuclear C*-algebra. The following are equivalent:

(i) $A \in \mathcal{N}$.

(ii) A is KK-equivalent to a commutative C*-algebra.

(iii) The sequence of I.4.6.15. is exact for every separable C*-algebra B.

The equivalence of (i) and (iii) justifies calling $\mathcal{N}$ the "UCT Class."

The Künneth Theorem for Tensor Products can be similarly proved:

I.4.6.19. Theorem. [Künneth Theorem for Tensor Products (KTP)] [Sch82] Let A and B be C*-algebras, with $A \in \mathcal{N}$. Then there is a short exact sequence

$$0 \longrightarrow K_*(A) \otimes_{\mathbb{Z}} K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow 0.$$

The map α has degree 0 and σ has degree 1. The sequence is natural in each variable, and splits unnaturally. So if $K_*(A)$ or $K_*(B)$ is torsion-free, α is an isomorphism.

This theorem allows efficient computation of the K-theory of a tensor product.

I.4.6.20. There is another similar theorem called the *Künneth Theorem*, but it would take us afield to even state it and it does not seem to be directly relevant to classification. See [Bla98].

Why Only Nuclear or UCT C*-Algebras?

I.4.6.21. Why do we limit ourselves to trying to classify only (separable) nuclear or UCT C*-algebras? To have any hope of classifying by K-theoretic invariants (e.g. the Elliott invariant), we must make this restriction, for a couple reasons:

(i) As claimed in [ET08], we cannot reasonably hope to give an effective classification of a class of C*-algebras unless we can give an effective description of which von Neumann algebras are generated by representations of these algebras (I do not entirely agree with this – it sounds a little too much like the pre-1970s claim that we can only "understand" Type I C*-algebras). If A is nuclear, then every representation of A generates an AFD von Neumann algebra, and conversely (). AFD von Neumann algebras are effectively classified, but no larger class is (yet).

(ii) Nuclear and even UCT examples exhaust the possibilities for the Elliott invariant, so classification via the Elliott invariant cannot be possible for any strictly larger class. Classification for nuclear C*-algebras will only be possible if every separable simple nuclear C*-algebra satisfies the UCT. While we do not know whether the class of separable nuclear C*-algebras is larger than the class of UCT algebras, we do know that the class of separable exact C*-algebras is strictly larger. To get a dramatic and concrete illustration, let G be the group of I.6.5.3.. If A is any separable simple unital nuclear (say UCT) C*-algebra with $K_0(A)$ weakly unperforated (e.g. of finite nuclear dimension), then $A \otimes C_r^*(G)$ is a separable simple unital C*-algebra with the same Elliott invariant as A, which is exact but nonnuclear. We can tensor on $\mathcal{Z}$, O_∞, or O_2 to get exact nonnuclear examples with seemingly all K-theoretic properties of standard UCT examples.

I.5. Purely Infinite C*-Algebras

There is a class of separable simple nuclear C*-algebras which are at the opposite extreme to the simple ASH algebras, the purely infinite (nuclear) C*-algebras. These are the C*-algebra analogs of the Type III factors.

I.5.1. Cuntz and Cuntz-Krieger Algebras

The first natural class of purely infinite separable nuclear C*-algebras was systematically studied by J. Cuntz in [Cun77a]. These algebras had been previously described, e.g. by Dixmier [Dix64], but even the most basic properties were not established before [Cun77a].

Purely Infinite C*-Algebras

I.5.1.1. Definition. A unital C*-algebra A is *purely infinite* if $A \ncong \mathbb{C}$ and for every $x \in A$, $x \neq 0$, there is an isometry s in A (i.e. $s^*s = 1$) with $ss^* \in x^*Ax$.

If $x \neq 0$, there is a $y \neq 0$ such that $\overline{y^*Ay} \subseteq x^*Ax$; thus the last condition can be replaced by the apparently weaker condition that $ss^* \in \overline{x^*Ax}$. Since $\overline{x^*Ax} = \overline{(x^*x)A(x^*x)}$, the x may be assumed positive. An equivalent formulation is that $A \ncong \mathbb{C}$ is purely infinite if for every $a \in A_+$, $a \neq 0$, there is a projection p equivalent to 1 and $\lambda > 0$ such that $\lambda p \leq a$. In a purely infinite C*-algebra, there is no notion of "size" of support of elements: the support of every nonzero positive element is of "infinite size."

I.5.1.2. Any countably decomposable Type III factor () is purely infinite. (This is not immediately obvious, but easily proved.)

I.5.1.3. Proposition. Let A be a unital C*-algebra, $A \ncong \mathbb{C}$. The following are equivalent:

(i) A is purely infinite.

(ii) For every $a \in A$, $a \neq 0$, there are $x, y \in A$ with $xay = 1$.

(iii) For every $a \in A$, $a \neq 0$, there are $x, y \in A$ with xay invertible.

(iv) For every $a \in A_+$, $a \neq 0$, there is an $x \in A$ with $x^*ax = 1$.

(v) For every $a \in A_+$, $a \neq 0$, there is an $x \in A$ with x^*ax invertible.

See () for more equivalent conditions.

Proof: (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are trivial; (iii) $\Rightarrow$ (ii) is essentially trivial, and (v) $\Rightarrow$ (iv) is easy: if x^*ax is invertible, then $[(x^*ax)^{-1/2}x^*]a[x(x^*ax)^{-1/2}] = 1$. (iv) $\Rightarrow$ (ii) is also essentially trivial: if $0 \neq a \in A$, then there is an $x \in A$ with $x^*(a^*a)x = 1$.
(ii) $\Rightarrow$ (v): Let $0 \neq a \in A_+$. There are $x, y \in A$ with $xa^{1/2}y = 1$. Then

$$1 = (xa^{1/2}y)^*(xa^{1/2}y) = x^*[a^{1/2}y^*ya^{1/2}]x \leq \|y\|^2 x^*ax$$

so x^*ax is invertible.

(i) $\Rightarrow$ (ii): Suppose A is purely infinite and $0 \neq a \in A$. Let s be an isometry in A with $ss^* = a^*ba$ for some $b \in A_+$. Then $1 = s^*ss^*s = (s^*a^*b)as$.
(iv) $\Rightarrow$ (i): Let $0 \neq a \in A_+$. there is an $x \in A$ with $x^*ax = 1$. If $s = a^{1/2}x$, then $s^*s = 1$, so s is an isometry, and

$$ss^* = a^{1/2}xx^*a^{1/2} \leq \|x\|^2 a \in \overline{aAa}$$

so A is purely infinite. 👍

I.5.1.4. Corollary. A corner in a purely infinite C*-algebra is purely infinite.

Proof: If A is purely infinite and p is a nonzero projection in A, and a is a nonzero element of pAp, there are x and y in A with $xay = 1$. Then $(pxp)a(pyp) = p$ in pAp. 👍

I.5.1.5. Proposition. A purely infinite C*-algebra is simple and properly infinite.

Proof: Simplicity follows immediately from I.5.1.3.: if I is a nonzero ideal in A and $a \in I$, and there are $x, y \in A$ with $xay = 1$, then $1 \in I$ so $I = A$.

A purely infinite C*-algebra is at least two-dimensional (in fact, infinite-dimensional), so contains a positive element whose spectrum has more than one element, hence by functional calculus contains two orthogonal nonzero positive elements a and b. There are projections p, q equivalent to 1 with $p \in aAa$, $q \in bAb$, so $p \perp q$; thus A is properly infinite. (Actually, any infinite simple C*-algebra is properly infinite (III.2.1.2.).) 👍

I.5.1.6. There are properly infinite C*-algebras which are not purely infinite, e.g. $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ infinite-dimensional. It is a subtle question whether there exist *simple* properly infinite C*-algebras which are not purely infinite (cf. ()).

I.5.1.7. A purely infinite C*-algebra has a lot of projections; in fact, it has real rank 0 (). If p and q are any nonzero projections in a purely infinite C*-algebra A, then there is a projection $p' \leq q$ which is equivalent to p (i.e. $p \precsim q$): let s be an isometry in A with $ss^* \in qAq$; then $p' = sps^* \leq ss^* \leq q$, and $u = sp$ is a partial isometry from p to p'.

A purely infinite C*-algebra A has cancellation of nonzero projections: if $r \perp p$, $r \perp q$, $p + r \sim q + r$, and $p \neq 0$, $q \neq 0$, then $p \sim q$ (). Since a matrix algebra over a purely infinite C*-algebra A is purely infinite (it is isomorphic to a corner in A), if p and q are nonzero projections in a matrix algebra over A in the same K_0-class, then $p \sim q$. But a purely infinite C*-algebra can have inequivalent nonzero projections (I.5.1.16.).

I.5.1.8. There is a reasonable way to extend the definition of "purely infinite" to nonsimple C*-algebras, although the proper definition is somewhat subtle; see [**?**]. We will not need this more general notion, so in this book "purely infinite" will mean the definition in I.5.1.1..

Cuntz Algebras

I.5.1.9. Definition. Let $n \geq 2$ be a natural number. The *Cuntz algebra* O_n is the universal unital C*-algebra generated by n isometries with mutually orthogonal range projections adding to the identity, i.e. O_n is the universal C*-algebra generated by elements $s_1, \ldots, s_n$ satisfying

$$s_k^* s_k = 1 \text{ for all } k$$

$$\sum_{k=1}^{n} s_k s_k^* = 1 \ .$$

The mutual orthogonality of the range projections $p_k = s_k s_k^*$ follows automatically from the second relation. The next theorem summarizes the most important basic properties of the O_n; the proof will be discussed below.

I.5.1.10. Theorem. The O_n are purely infinite, hence simple and properly infinite, and are nuclear (in fact, in the small bootstrap class).

Actually, O_n is transparently properly infinite from its definition. (In fact, any infinite simple unital C*-algebra is properly infinite (III.2.1.2.).)

I.5.1.11. The simplicity of O_n implies that if $s_1, \ldots, s_n$ is any set of n isometries in any unital C*-algebra A which satisfy the relations of I.5.1.9., then $\{s_1, \ldots, s_n\}$ generate a (unital) C*-subalgebra of A isomorphic to O_n.

There are some related nonsimple C*-algebras which are important:

I.5.1.12. Definition. Let $n \geq 1$ be a natural number. The C*-algebra T_n is the universal unital C*-algebra generated by n isometries with mutually orthogonal range projections, i.e. T_n is the universal C*-algebra generated by elements $s_1, \ldots, s_n$ satisfying

$$s_k^* s_k = 1 \text{ for all } k$$

$$s_j s_j^* \perp s_k s_k^* \text{ for } j \neq k \ .$$

I.5.1.13. T_n is also defined for $n = 1$. T_1, often denoted T, is called the *Toeplitz algebra*, the universal (unital) C*-algebra generated by one isometry. We could also define O_1 to be the universal C*-algebra generated by one unitary, i.e. $O_1 \cong C(\mathbb{T})$; we will generally exclude this algebra from consideration among the Cuntz algebras since it has quite different properties.

I.5.1.14. There is a natural quotient map from T_n ($n \geq 1$) to O_n, since the generators of O_n satisfy the same relations plus one additional one. The kernel of this quotient map is the ideal of T_n generated by $p = 1 - \sum_{k=1}^{n} s_k s_k^*$. It is not hard to see that this ideal is isomorphic to $\mathbb{K}$; thus there is an exact sequence

$$0 \to \mathbb{K} \to T_n \to O_n \to 0$$

which can be used to carefully calculate the K-theory of O_n ():

I.5.1.15. THEOREM. $K_0(O_n) = \mathbb{Z}_{n-1}$ and $K_1(O_n) = 0$. $[1]$ is a generator for $K_0(O_n)$.

To see the plausibility of this calculation (at least the K_0 part), note that

$$[1] = [p_1] = \cdots = [p_n] = [p_1 + \cdots + p_n] = [p_1] + \cdots + [p_n] = n[1]$$

in $K_0(A)$, so $(n-1)[1] = 0$.

I.5.1.16. This result shows a huge difference between the O_n ($n \geq 3$) and Type III factors. In a countably decomposable (simple) Type III factor, all nonzero projections are equivalent, so K_0 is trivial. But in O_n, $n \geq 3$, although there is no notion of one projection having "larger" support than another ($p \precsim q$ for any nonzero projections p and q), there are inequivalent nonzero projections: for example, $p_1 + p_2$ is not equivalent to 1 (or to p_1). In fact, K-theory is irrelevant in the classification of Type III AFD factors, but is the whole story for the classification of purely infinite separable nuclear C*-algebras. Only O_2 behaves like a Type III factor in that all nonzero projections are equivalent (and the unitary group is connected).

I.5.1.17. **Exercise:** (i) Let $n, m \in \mathbb{N}$. Show that there is a unital embedding of O_{n+1} into O_{m+1} if and only if m divides n. Give an explicit embedding in this case. [Hint: to embed O_3 into O_2, consider the elements s_1, $s_2 s_1$, and s_2^2 of O_2. There are other possibilities.]

(ii) Let $n, k \in \mathbb{N}$. Show there is a unital embedding of O_{n+1} into $M_k(O_{kn+1})$. [Hint: to embed O_3 into $M_2(O_5)$, consider the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & s_1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ s_2 & s_3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ s_4 & s_5 \end{bmatrix}$$

(there are many other possibilities).]

(iii) Let $n, m, k \in \mathbb{N}$. Give necessary conditions for existence of a unital embedding of O_{n+1} into $M_k(O_{m+1})$ (The right set of necessary conditions will also be sufficient; cf. I.5.1.24. and ().)

There is one more Cuntz algebra, rather different than the others but extremely important in the general theory:

I.5.1.18. DEFINITION. The *Cuntz algebra* O_∞ is the universal unital C*-algebra generated by a sequence of isometries with mutually orthogonal range projections, i.e. O_∞ is the universal C*-algebra generated by elements $s_1, s_2, \ldots$ satisfying

$$s_k^* s_k = 1 \text{ for all } k$$

$$s_j s_j^* \perp s_k s_k^* \text{ for } j \neq k \ .$$

O_∞ might more properly be called T_∞ since it is more analogous to the T_n than the O_n in its definition; but perhaps surprisingly it turns out to be more similar to the O_n in its properties.

I.5.1.19. THEOREM. O_∞ is purely infinite, hence simple and properly infinite (obvious from its definition).

I.5.1.20. Theorem. $K_0(O_\infty) = \mathbb{Z}$ and $K_1(O_\infty) = 0$. $[1]$ is a generator for $K_0(O_\infty)$.

See () for a proof.

I.5.1.21. The two most generally important Cuntz algebras are O_2 and O_∞; in fact, these are two of the most important C*-algebras of any kind. Among many other things, O_2 is important since its K-theory is trivial, and O_∞ is important since its K-theory is the same as the K-theory of $\mathbb{C}$. In fact, O_2 is KK-equivalent to the zero C*-algebra (*K-contractible*), and O_∞ is KK-equivalent to $\mathbb{C}$.

I.5.1.22. The Cuntz algebras, like all infinite simple C*-algebras, have no traces, and the positive cone of K_0 is the whole group (III.2.1.6.), so the Elliott invariant reduces to the triple $(K_0, K_1, [1])$, or abstractly (G_0, G_1, u), where G_0 and G_1 are countable abelian groups and $u \in G_0$. We will see that any such triple occurs for a unique purely infinite separable nuclear C*-algebra (satisfying the UCT). We have

$$Ell(O_n) = (\mathbb{Z}_{n-1}, 0, 1) \ (2 \leq n < \infty)$$

$$Ell(O_\infty) = (\mathbb{Z}, 0, 1)$$

I.5.1.23. If $s_1, \dots, s_n$ are the standard generators of O_n, we write L_n for the dense *-subalgebra of O_n generated algebraically by $\{s_1, \dots, s_n\}$, i.e. the set of finite sums of scalar multiples of (finite) words in the s_k and s_k^*. L_n is a *Leavitt path algebra* (). Other purely infinite C*-algebras have naturally associated Leavitt path algebras or similar dense *-subalgebras (for example, O_∞ has a natural dense *-subalgebra L_∞ generated algebraically by $\{s_1, s_2, \dots\}$), and there is a purely algebraic classification theory for such algebras which is parallel to the classification of purely infinite separable nuclear C*-algebras. However, the theories turn out to be fundamentally different in some respects: isomorphism of the C*-algebras does not imply isomorphism of the dense *-subalgebras in general, and the algebraic classification of the dense *-subalgebras is much more complicated (after the fact!) See [Abr15] and [AASM17] for a comprehensive survey and treatment.

I.5.1.24. The Elliott invariant of $M_k(O_n)$ is $(\mathbb{Z}_{n-1}, 0, k)$. Thus $M_k(O_n)$, which is purely infinite, cannot be isomorphic to O_n unless k and $n-1$ are relatively prime. If k and $n-1$ are relatively prime, e.g. if $k = n$, is $M_k(O_n)$ isomorphic to O_n? This is a tricky question in general, and was open for a long time. The answer is yes if k divides a power of n (and in particular if $k = n$): it is fairly easy to give an explicit isomorphism between $M_k(L_n)$ and L_n, and hence $M_k(O_n) \cong O_n$. It is much harder to settle the case where k does not divide a power of n. It was first shown using the Kirchberg-Phillips classification () that $M_k(O_n) \cong O_n$ whenever k and $n-1$ are relatively prime, but this was thought to be a deep fact; to the surprise of many, an explicit isomorphism of $M_k(L_n)$ and L_n in the general (relatively prime) case was given in 2008 [AAP08], which extends to an isomorphism $M_k(O_n) \cong O_n$.

I.5.1.25. **Exercise:** (i) Give an explicit isomorphism of $M_n(L_n)$ with L_n for any n (easy).

(ii) Give an explicit isomorphism of $M_2(L_4)$ with L_4 (slightly harder). Generalize to give an explicit isomorphism of $M_k(L_n)$ with L_n if $k|n$.

(iii) Observing that if $M_j(L_n) \cong M_k(L_n) \cong L_n$, then $M_{jk}(L_n) \cong M_j(M_k(L_n)) \cong L_n$, show how to give an explicit isomorphism of $M_k(L_n)$ with L_n if k divides a power of n (i.e. every prime factor of k divides n).

(iv) Give an explicit isomorphism of $M_3(L_5)$ with L_5 (pretty challenging; cf. [Abr15]).

Structure of the Cuntz Algebras

Since full proofs of the properties are available in several references, e.g. (), we will merely outline the results.

I.5.1.26. The key to understanding the structure of the Cuntz algebra O_n and proving its properties is to note that since $s_j^* s_k = \delta_{jk} 1$ (Kronecker δ), the elements $s_j s_k^*$ $(1 \le j, k \le n)$ form a system of $n \times n$ matrix units in O_n (), hence generate a unital copy of $\mathbb{M}_n$. The p_k are the minimal diagonal projections of this matrix algebra. More generally, every element of the dense subalgebra L_n is a finite linear combination of words of the form $s_\mu s_\nu^*$, where μ and ν denote multi-indices (we allow the empty multi-index, whose corresponding word is 1). We have $\|s_\mu s_\nu^*\| = 1$ for all μ and ν, and it follows from the simplicity of matrix algebras that the $s_\mu s_\nu^*$ are linearly independent in O_n. Write $|\mu|$ for the length of μ. The closed linear span of the $s_\mu s_\nu^*$ with $|\mu| = |\nu|$ is a unital C*-subalgebra F_n of O_n which is a UHF algebra of type n^∞. The map $x \mapsto s_k x s_k^*$ $(1 \le k \le n)$ is a nonunital endomorphism of F_n onto the corner $p_k F_n p_k$.

I.5.1.27. It follows readily that if we form the infinite tensor product (I.1.7.17.)

$$A = \bigotimes_{k \in \mathbb{Z}} (\mathbb{M}_n, q_k)$$

where $q_k = 1$ for $k \ge 0$ and q_k is a rank-one projection for $k < 0$, then $A \cong F_n \otimes \mathbb{K}$, and if σ is the shift on the tensor product factors, then

$$A \rtimes_\alpha \mathbb{Z} \cong O_n \otimes \mathbb{K}\ .$$

Thus $O_n \otimes \mathbb{K}$ is the crossed product of an AF algebra by $\mathbb{Z}$, hence in the small bootstrap class and in particular nuclear.

I.5.1.28. There is a conditional expectation (completely positive projection of norm 1) from O_n onto the C*-subalgebra F_n, defined as follows. Note that if $\lambda \in \mathbb{C}$, $|\lambda| = 1$, there is an automorphism α_λ of O_n sending s_k to λs_k for each k. The α_λ define a point-norm continuous action of the circle group $\mathbb{T}$ on O_n (called the *gauge action*). For $x \in O_n$, define

$$\theta(x) = \int_{\mathbb{T}} \alpha_\lambda(x)\, d\lambda$$

using Haar measure on the circle (Lebesgue measure) normalized to have total mass 1. We clearly have $\|\theta(x)\| \le \|x\|$ for any x, hence $\|\theta\| \le 1$. A simple calculation shows that

$$\theta(s_\mu s_\nu^*) = \begin{cases} s_\mu s_\nu^* & \text{if} \quad |\mu| = |\nu| \\ 0 & \text{if} \quad |\mu| \ne |\nu| \end{cases} .$$

It follows that $\theta(x) \in F_n$ for all $x \in L_n$, and hence by continuity that $\theta(x) \in F_n$ for all $x \in O_n$; and $\theta(x) = x$ for $x \in F_n$. (Thus in particular $\|\theta\| = 1$.) Complete positivity of θ is obvious.

I.5.1.29. PROPOSITION. If $0 \ne x \in O_n$, then $\theta(x^* x) \ne 0$ (θ is *faithful*).

This can be proved by direct calculation, or it follows from a general fact about crossed products by discrete abelian groups (the essential fact is that if G is a discrete abelian group, the characters of $\hat{G}$ come from elements of G and form an orthonormal basis for $L^2(\hat{G})$). Note that the "proof" in [**?**] is not adequate.

I.5.1.30. A proof that O_n is purely infinite follows. If $0 \neq x \in O_n$ and $a = x^*x$, then $\|\theta(a)\| > 0$; we may assume $\|\theta(a)\| = 1$. Choose $z \in L_n$ with $\|y - a\| < \frac{1}{4}$, where $y = z^*z$; then $\|\theta(y)\| > \frac{3}{4}$. One constructs a projection q in the dense locally finite subalgebra of F_n such that $\|q\theta(y)q\| = \|\theta(y)\|$ and $q\theta(y)q = qyq$. Then q and $q\theta(y)q$ lie in a matrix subalgebra M of F_n isomorphic to $\mathbb{M}_{n^r}$ for some r; there is a minimal projection p in M with $p \leq q$ and $\|p\theta(y)p\| = \|pq\theta(y)qp\| > \frac{3}{4}$. Then $p \sim 1$ in O_n (even in L_n), so there is an isometry $u \in L_n$ with $uu^* = p$. We have $u^*yu = u^*\theta(y)u = \lambda 1$ for some λ, $|\lambda - 1| < \frac{1}{4}$, and thus

$$\|u^*au - 1\| \leq \|u^*au - u^*yu\| + \|u^*yu - 1\| < \frac{1}{2}$$

and u^*au is invertible. Thus $[(u^*au)^{-1}u^*x^*]xu = 1$.

I.5.1.31. The argument for O_∞ is similar. The C*-subalgebra F_∞ generated by the $s_\mu s_\nu^*$ is not a UHF algebra in this case, but it is an AF algebra, and the arguments for O_n carry over almost verbatim with a few additional technicalities.

Cuntz-Krieger Algebras

A more general class of purely infinite C*-algebras was considered by Cuntz and Krieger [CK80].

I.5.1.32. Definition. Let $A = [\alpha_{jk}]$ be an $n \times n$ matrix of 0's and 1's ($n > 1$), with no zero rows or columns. The *Cuntz-Krieger* algebra O_A is the universal C*-algebra generated by n partial isometries $s_1, \ldots, s_n$ with mutually orthogonal range projections $p_k = s_k s_k^*$, satisfying

$$s_j^* s_j = \sum_{k=1}^{n} \alpha_{jk} s_k s_k^* = \sum_{k=1}^{n} \alpha_{jk} p_k$$

for all j.

I.5.1.33. A Cuntz-Krieger algebra is unital, with unit $\sum_{k=1}^n p_k$. O_n is the special case where A is the $n \times n$ matrix of all ones.

A Cuntz-Krieger algebra is not necessarily simple. For example, if A is the identity matrix, then O_A is a direct sum of n copies of $C(\mathbb{T})$. There is a clean characterization of simplicity:

I.5.1.34. Theorem. Let A be an $n \times n$ matrix of 0's and 1's ($n > 1$), with no zero rows or columns. The following are equivalent:

(i) O_A is simple.

(ii) O_A is purely infinite.

(iii) A is irreducible (I.5.1.35.) and not a permutation matrix.

(iv) There is an m such that every entry of A^m is strictly positive.

(We will call such an A *simple*; the name *aperiodic* is also sometimes used.)

The proof is very similar to the argument in the case of O_n; the aperiodicity condition is easily seen to be necessary for simplicity.

I.5.1.35. These algebras can be described using directed graphs. Let G be a directed graph with $n > 1$ vertices $v_1, \dots, v_n$, with no sources or sinks (every vertex has at least one edge in and one edge out), and such that for any vertices v_j and v_k there is at most one edge from v_j to v_k. For each j and k, set $\alpha_{jk} = 1$ if there is an edge from v_k to v_j (note the conventional order of j and k!) and 0 otherwise. One thus obtains an $n \times n$ matrix of 0's and 1's with no zero rows or columns. It is clear that every such matrix arises from such a graph. The matrix A is by definition irreducible if the graph is *connected*, i.e. for any j and k there is a path from v_j to v_k.

I.5.1.36. The O_A also arise naturally in the study of topological Markov chains ().

I.5.1.37. Theorem. Let A be an $n \times n$ matrix of 0's and 1's ($n > 1$), with no zero rows or columns. Then O_A is separable and in the small bootstrap class, hence nuclear.

It is obvious that a Cuntz-Krieger algebra is separable, since it is finitely generated. The last statement is not so obvious, but easily proved ().

The K-theory of O_A can be easily calculated. The matrix A gives an endomorphism of $\mathbb{Z}^n$, as does $I - A$, where I is the $n \times n$ identity matrix.

I.5.1.38. Theorem. Let A be an $n \times n$ matrix of 0's and 1's ($n > 1$), with no zero rows or columns. Then the K-theory of O_A is

$$K_0(O_A) \cong \mathbb{Z}^n/(I - A)\mathbb{Z}^n$$
$$K_1(O_A) \cong \ker(I - A).$$

The order unit [1] in $K_0(O_A)$ corresponds to the class of $(1, 1, \dots, 1)$ in $\mathbb{Z}^n/(I - A)\mathbb{Z}^n$.

This result applies whether or not O_A is simple.

I.5.1.39. The K_0 and K_1 of an O_A are not independent: $K_1(O_A)$ is always exactly the torsion-free part of $K_0(O_A)$. If G is any finitely generated abelian group, for some n there is an irreducible $n \times n$ matrix A for which $\mathbb{Z}^n/(I - A)\mathbb{Z}^n \cong G$ (). Thus we obtain:

I.5.1.40. Corollary. Let G_0 be any finitely generated abelian group, and G_1 the torsion-free part of G_0. Then there is a simple Cuntz-Krieger algebra O_A with $K_0(O_A) \cong G_0$, $K_1(O_A) \cong G_1$. In particular, if G_0 is any finitely generated abelian group and G_1 the torsion-free part of G_0, and u is any element of G_0, there is a purely infinite separable simple nuclear C*-algebra with Elliott invariant (G_0, G_1, u) (a corner in a Cuntz-Krieger algebra).

I.5.1.41. Example. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then A is simple, so O_A is purely infinite. It is easy to check from I.5.1.38. that $K_0(O_A) \cong K_1(O_A) \cong \mathbb{Z}$. This O_A is sometimes called O_{1^-}.

I.5.1.42. Example. More generally, let A be an $n \times n$ matrix of 0's and 1's, and let B be the $(n+2) \times (n+2)$ matrix

$$\begin{bmatrix} & & & 0 & 0 \\ & A & & \vdots & \vdots \\ & & & 1 & 0 \\ 0 & \cdots & 1 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

with two additional rows and columns at the end with six 1's in the lower corner. It is straightforward to check that if A is simple, then so is B, and that $K_0(O_B) \cong K_0(O_A)$ and hence $K_1(O_B) \cong K_1(O_A)$. The algebra O_B is sometimes called O_{A^-}; the reason is that $det(I_{n+2} - B) = -det(I_n - A)$. It was mysterious for quite some time whether $O_A \cong O_{A^-}$ (if A is simple); they turn out to be isomorphic (using the classification theorem), but the canonical dense subalgebras are not isomorphic, so the isomorphism cannot be explicitly described algebraically. (It is straightforward to show that $O_{(A^-)^-} \cong O_A$, and in fact the dense subalgebras are isomorphic.)

Tensor Products

I.5.1.43. Many more examples can be made via tensor products. A tensor product of two separable simple unital nuclear C*-algebras is again separable, simple, unital, and nuclear. If both factors have a trace, so does the tensor product. If one factor is properly infinite, so is the product. If one or even both factors are purely infinite, it is not obvious that the product is purely infinite, but it turns out to be true if the factors are classifiable. Since the class of purely infinite C*-algebras is closed under taking matrix algebras and inductive limits, it follows that the tensor product of a purely infinite C*-algebra with a UHF algebra is purely infinite, and it is not much harder to show that the tensor product of a purely infinite C*-algebra with a simple AF algebra is purely infinite. It is somewhat harder, although still not difficult, to show that the tensor product of a purely infinite C*-algebra with a simple A$\mathbb{T}$ algebra of real rank zero is purely infinite.

I.5.1.44. If B is a unital AF algebra, and A is any unital C*-algebra, it is elementary to show that

$$K_0(A \otimes B) \cong K_0(A) \otimes_{\mathbb{Z}} K_0(B) \quad \text{and} \quad K_1(A \otimes B) \cong K_1(A) \otimes_{\mathbb{Z}} K_0(B)$$

(the "unital" is superfluous here). It is also straightforward to show using some elementary K-theory that

$$K_0(A \otimes C(\mathbb{T})) \cong K_1(A \otimes C(\mathbb{T})) \cong K_0(A) \oplus K_1(A)$$

for any unital C*-algebra A ("unital" is again superfluous). It then follows that if B is an A$\mathbb{T}$ algebra, then for any C*-algebra A we have

$$K_0(A \otimes B) \cong [K_0(A) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_1(A) \otimes_{\mathbb{Z}} K_1(B)]$$

$$K_1(A \otimes B) \cong [K_1(A) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_0(A) \otimes_{\mathbb{Z}} K_1(B)]$$

by a limit calculation. The K-theory of any tensor product of classifiable C*-algebras can be calculated using the Künneth Theorem for Tensor Products (I.4.6.19.); the formula is similar, but more complicated if there is torsion in both K-theories.

I.5.1.45. EXAMPLES. Fix an $n > 2$.

(i) Let B be the Bunce-Deddens algebra (I.3.6.13.) with supernatural number $(n-1)^\infty$, and set $A = O_n \otimes B$. Then A is purely infinite (I.5.1.43.) and nuclear. By I.5.1.44., we have

$$K_0(A) \cong [K_0(O_n) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_1(O_n) \otimes_{\mathbb{Z}} K_1(B)] = [\mathbb{Z}_{n-1} \otimes_{\mathbb{Z}} K_0(B)] \oplus [0 \otimes_{\mathbb{Z}} \mathbb{Z}] = 0$$

$$K_1(A) \cong [K_1(O_n) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_0(O_n) \otimes_{\mathbb{Z}} K_1(B)] = [0 \otimes_{\mathbb{Z}} K_0(B)] \oplus [\mathbb{Z}_{n-1} \otimes_{\mathbb{Z}} \mathbb{Z}] \cong \mathbb{Z}_{n-1} \ .$$

This A is sometimes called P_n.

(ii) Let D be the Bunce-Deddens algebra with supernatural number n^∞, and set $A = O_n \otimes D$. Then A is purely infinite (I.5.1.43.) and nuclear. By I.5.1.44., we have

$$K_0(A) \cong [K_0(O_n) \otimes_{\mathbb{Z}} K_0(D)] \oplus [K_1(O_n) \otimes_{\mathbb{Z}} K_1(D)] = [\mathbb{Z}_{n-1} \otimes_{\mathbb{Z}} K_0(D)] \oplus [0 \otimes_{\mathbb{Z}} \mathbb{Z}] \cong \mathbb{Z}_{n-1}$$

$$K_1(A) \cong [K_1(O_n) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_0(O_n) \otimes_{\mathbb{Z}} K_1(B)] = [0 \otimes_{\mathbb{Z}} K_0(B)] \oplus [\mathbb{Z}_{n-1} \otimes_{\mathbb{Z}} \mathbb{Z}] \cong \mathbb{Z}_{n-1} \ .$$

This A turns out to be isomorphic to $O_n \otimes O_n$, although classification is needed to prove the isomorphism. Even showing that $O_n \otimes O_n$ is purely infinite is quite nontrivial, and computing the K-theory of $O_n \otimes O_n$ requires some version of the the Künneth Theorem for Tensor Products.

(iii) Let B be a simple A$\mathbb{T}$ algebra of real rank zero with $K_0(B)$ the triadic rationals $\mathbb{Z}\left[\frac{1}{3}\right]$ and $K_1(B)$ the dyadic rationals $\mathbb{Z}\left[\frac{1}{2}\right]$, and set $A = O_7 \otimes B$. Then A is purely infinite (I.5.1.43.) and nuclear. By I.5.1.44., we have

$$K_0(A) \cong [K_0(O_7) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_1(O_7) \otimes_{\mathbb{Z}} K_1(B)] = \left[\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{3}\right]\right] \oplus \left[0 \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right]\right] \cong \mathbb{Z}_2$$

$$K_1(A) \cong [K_1(O_7) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_0(O_7) \otimes_{\mathbb{Z}} K_1(B)] = \left[0 \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{3}\right]\right] \oplus \left[\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right]\right] \cong \mathbb{Z}_3 \ .$$

In a similar way, if p, q are any natural numbers greater than 1, a purely infinite A can be constructed with $K_0(A) = \mathbb{Z}_p$ and $K_1(A) = \mathbb{Z}_q$ (it is a little harder if p and q are not relatively prime or equal).

(iv) Let θ be an irrational number in $[0,1]$, let $B = \mathbb{M}_\theta$ (I.3.2.12.), and set $A = O_n \otimes B$. Then A is purely infinite (I.5.1.43.) and nuclear. By I.5.1.44., we have

$$K_0(A) \cong [K_0(O_n) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_1(O_n) \otimes_{\mathbb{Z}} K_1(B)] = [\mathbb{Z}_{n-1} \otimes_{\mathbb{Z}} \mathbb{Z}^2] \oplus [0 \otimes_{\mathbb{Z}} \mathbb{Z}] = \mathbb{Z}_{n-1} \oplus \mathbb{Z}_{n-1}$$

$$K_1(A) \cong [K_1(O_n) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_0(O_n) \otimes_{\mathbb{Z}} K_1(B)] = [0 \otimes_{\mathbb{Z}} K_0(B)] \oplus [\mathbb{Z}_{n-1} \otimes_{\mathbb{Z}} 0] = 0 \ .$$

It is not obvious that the tensor products of O_n with $\mathbb{M}_\theta$'s for different θ are isomorphic, but they turn out to be.

(iv) Tensoring the A of (iv) with suitable Bunce-Deddens algebras gives more examples; tensoring O_n's or O_A's with various simple A$\mathbb{T}$ algebras of real rank zero give many more examples. One could also tensor O_n with an irrational rotation algebra to obtain a purely infinite A with $K_0(A) = K_1(A) = \mathbb{Z}_{n-1} \oplus \mathbb{Z}_{n-1}$, the same K-theory as the tensor product of the A in (iv) with the D of (ii). It is again not at all obvious that the tensor products are isomorphic, irrespective of the irrational angle chosen, but they are.

The reader is urged to see how many different K-theories can be obtained in this manner.

More examples can be made by taking inductive limits:

I.5.1.46. Proposition. An inductive limit of purely infinite C*-algebras is purely infinite.

Proof: This almost has a very easy proof. Let $A = \lim_{\rightarrow} A_n$, where each A_n is purely infinite (and the connecting maps are unital). Let $0 \neq a \in A$. Choose b in some A_n closely approximating a. Then there are $x, y \in A_n$ with $xby = 1$. Then $xay \approx 1$, hence xay is invertible. However, this does not directly work since at the moment we have no control over $\|x\|$ and $\|y\|$. These norms can be controlled (); but we avoid the problem with a little functional calculus.

Let f and g be continuous functions from $[0,1]$ to $[0,1]$ with $f(0) = 0$, f identically 1 on $\left[\frac{1}{2}, 1\right]$, g identically 0 on $\left[0, \frac{1}{2}\right]$, and $g(1) = 1$. Then $fg = g$. Let $a \in A_+$, $\|a\| = 1$. Choose $b \in (A_n)_+$ for some n with $\|b\| = 1$ and b so close to a that $\|f(b) - f(a)\| < 1$ (this can be done since f can be uniformly approximated by polynomials; cf. ()). Then $0 \neq g(b) \in (A_n)_+$, so there is an $x \in A_n$ with $x^*g(b)x = 1$. Set $y = g(b)^{1/2}x$; then y is an isometry and $f(b)y = y$. We have

$$\|y^*f(a)y - 1\| = \|y^*[f(a) - f(b)]y\| < 1$$

since $\|y\| = 1$; thus $y^*f(a)y$ is invertible. If $z = f(a)^{1/2}y(y^*f(a)y)^{-1/2}$, then $z^*z = 1$ and $zz^* \in f(a)^{1/2}Af(a)^{1/2} \subseteq \overline{aAa}$. 👍

In fact, any countable abelian groups can be obtained as the K-groups of a UCT Kirchberg algebra by an extension of the above constructions (IV.4.2.8.)

I.5.2. Kirchberg Algebras

By the early 1990s, there had been an impressive string of classification and structure results about stably finite separable simple nuclear C*-algebras, and there was a feeling we were on our way to a full understanding of them although much remained to be done; but, although there was a good supply of examples in the purely infinite case, classification in this case was still a complete mystery (even the matrix question for the O_n (I.5.1.24.) was not yet solved). Then M. Rørdam, partly in collaboration with Elliott, made some substantial progress in the classification of purely infinite C*-algebras [Rør93], [Rør95], [ER95], [Rør97], although most of his results still had an assumption that the algebra was an inductive limit of a certain kind (cf. I.6.6.1.(iv)). Soon after, a remarkable string of results of E. Kirchberg appeared which quickly led to a complete classification in the purely infinite case (carried out by Kirchberg and independently by N. C. Phillips using Kirchberg's announced results), which put the purely infinite case to rest except for the UCT problem, far in advance of the stably finite case. Kirchberg's results were not published in a straightforward way; see [Kir95] for an announcement without proofs, [KP00b] and [KP00c] for some of the key technical results, and [RS02] for a full exposition. Phillips' version of the classification is in [Phi00].

I.6. Trouble in Paradise

In the heady days of the mid-1990s, after the classification results described up to now were obtained, conventional wisdom began to swing from extreme pessimism that the classification program could get very far to wild optimism that perhaps the Elliott invariant indeed tells the whole story in the nuclear case. Many people started believing that "simple C*-algebras really are simple." But there were clouds on the horizon, and the storm soon hit which dashed this hope.

I.6.1. Cancellation, Strict Comparison, and Slow Dimension Growth

The first note of caution had been raised as early as 1991 in [Bla93] (compare [Bla93] with [Bla88] to see how the author's thinking changed between 1987 and 1991). Pathologies in matrix algebras over commutative C*-algebras could potentially persist in simple AH algebras if one is not careful to control the dimensions of the base spaces, and these pathologies could make the Elliott invariant inadequate for classification.

Cancellation

I.6.1.1. Definition. Let A be a C*-algebra. A has *cancellation of projections* if, whenever p, q, r are projections in A with $r \perp p$, $r \perp q$, $p+r \sim q+r$, we have $p \sim q$. A has *cancellation* if $M_n(A)$ has cancellation of projections for all n (equivalently, $A \otimes \mathbb{K}$ has cancellation of projections).

A unital C*-algebra A has cancellation if and only if the semigroup $V(A)$ () has cancellation, or equivalently the map from $V(A)$ to $K_0(A)$ is injective, i.e. two projections in matrix algebras over A are equivalent if and only if they have the same K_0-class.

I.6.1.2. If A contains an infinite projection, then A cannot have cancellation of projections: if $p \sim q \leq p$, $q \neq p$, then $0 + q \sim (p - q) + q$, but $0 \not\sim p - q$. Thus a (unital) C*-algebra with cancellation must be stably finite.

I.6.1.3. However, a stably finite C*-algebra does not necessarily have cancellation. For example, matrix algebras over commutative C*-algebras often fail to have cancellation of projections (see () for matrix algebras over $C(\mathbb{T}^5)$). In vector bundle language, if V_1, V_2, W are complex vector bundles over a compact Hausdorff space X and $V_1 \oplus W \cong V_2 \oplus W$ (Whitney sum), it is not necessarily true that $V_1 \cong V_2$.

I.6.1.4. It is obvious that a finite direct sum of C*-algebras has cancellation of projections [cancellation] if and only if each summand has cancellation of projections [cancellation].

Strict Comparison

Cancellation and Comparison for Vector Bundles

There are stability results for vector bundles, which can be found e.g. in [Hus94]. Fix a finite CW-complex X, and set

$$m = \left\lceil \frac{dim(X) + 1}{2} \right\rceil - 1$$

$$m' = \left\lceil \frac{dim(X)}{2} \right\rceil$$

where $\lceil \cdot \rceil$ denotes "least integer $\geq$" (the denominator of 2 arises since we are considering complex vector bundles and $\mathbb{C}$ is topologically two-dimensional). Note that $m \leq m' \leq m+1$, and $m + m' = dim(X)$.

In the following, we will take all vector bundles to have constant rank. More general vector bundles can be reduced to this case by partitioning X into clopen subsets.

I.6.1.5. THEOREM. [Hus94, 8.1.2] Let V be a k-dimensional complex vector bundle over X, with $k > m$. Then $V \cong W \oplus T_{k-m}$, where W is m-dimensional and T_{k-m} is a $(k-m)$-dimensional trivial bundle.

I.6.1.6. THEOREM. [Hus94, 8.1.5] Let V and W be complex vector bundles over X of rank $k \geq m'$. If $V \oplus T \cong W \oplus T$ for some trivial bundle T, then $V \cong W$.

The following corollary is not stated in [Hus94], but is immediate:

I.6.1.7. COROLLARY. Let V_1, V_2, W be complex vector bundles over X with $V_1 \oplus W \cong V_2 \oplus W$ (thus V_1 and V_2 necessarily have the same rank). If $rank(V_1) \geq m'$, then $V_1 \cong V_2$.

PROOF: Every complex vector bundle over X, and in particular W, is a direct summand of a trivial bundle ().

We can be more precise about how large a bundle must be added to give a trivial bundle:

I.6.1.8. COROLLARY. Let V be a complex vector bundle over X. Then there is a complex vector bundle W of rank m such that $V \oplus W$ is a trivial bundle.

PROOF: There is some bundle W' so that $V \oplus W' \cong T_n$, a trivial bundle of rank n, for some n which we can take to be at least $r = m + rank(V)$. By I.6.1.5. there is a W of rank m such that $W' \cong W \oplus T_{n-r}$. We have

$$V \oplus W \oplus T_{n-r} \cong V \oplus W' \cong T_n \cong T_r \oplus T_{n-r}$$

and since $r \geq m'$, we conclude $V \oplus W \cong T_r$ by I.6.1.6..

I.6.1.9. THEOREM. Let X be a finite CW-complex, and V_1, V_2 complex vector bundles over X. If $rank(V_2) - rank(V_1) \geq 2m$, where m is as in I.6.1.3., then V_1 is isomorphic to a subbundle of V_2.

PROOF: Set $r = rank(V_1)$ and $s = rank(V_2)$. By I.6.1.8. there is a bundle W_1 of rank m such that $V_1 \oplus W_1 \cong T_{r+m}$. By I.6.1.5. there is a bundle W_2 of rank m such that

$$V_2 \cong W_2 \oplus T_{s-m} \cong W_2 \oplus T_{r+m} \oplus T_{(s-m)-(r+m)} \cong W_2 \oplus V_1 \oplus W_1 \oplus T_{s-r-2m}$$

and thus V_1 is isomorphic to a summand of V_2.

Note that $2m = dim(X)$ if $dim(X)$ is even, and $2m = dim(X) - 1$ if $dim(X)$ is odd. So in particular, if $rank(V_2) - rank(V_1) \geq dim(X)$, the hypotheses are satisfied.

Rephrasing in terms of projections, we obtain:

I.6.1.10. COROLLARY. Let p, q, r be projections in a matrix algebra over $C(X)$ with $r \perp p$, $r \perp q$, $p+r \sim q+r$ (thus p and q necessarily have the same rank). If $rank(p) \geq m'$, where m' is as in I.6.1.3., then $p \sim q$.

I.6.1.11. THEOREM. Let X be a finite CW-complex, and p, q projections in a matrix algebra over $C(X)$. If $rank(q) - rank(p) \geq 2m$, where m is as in I.6.1.3. (in particular, if $rank(q) - rank(p) \geq dim(X)$), then $p \precsim q$.

I.6.1.12. If X is connected, then for any projection p in $M_k(C(X))$ and any trace τ on $M_k(C(X))$, $\tau(p)$ is exactly the normalized rank $\frac{rank(p)}{k}$ of p. Thus we may rephrase I.6.1.11.:

I.6.1.13. COROLLARY. Let X be a connected finite CW-complex, and p, q projections in $M_k(C(X))$. If $\tau(q) - \tau(p) \geq \frac{dim(X)}{k}$ for some (hence any) trace τ on $M_k(C(X))$, then $p \precsim q$.

I.6.1.10. can be similarly rephrased in terms of traces: projections of sufficiently large trace which are stably equivalent are equivalent.

I.6.1.14. The above results are stated and proved for finite CW-complexes. But routine limit theorems imply they hold also if X is any compact metrizable space (of finite topological dimension).

Slow Dimension Growth

There is a natural condition on AH inductive systems which will insure that the possible pathologies in the building blocks disappear and that the inductive limit has cancellation and strict comparison. Roughly speaking, the requirement is that as we go out the system, the sizes of the matrix algebras grow faster than the dimensions of the base spaces. This condition was introduced in [BDR91].

I.6.1.15. To illustrate, suppose the inductive system consists of a single matrix algebra at each level, i.e. $A_n = M_{k_n}(C(X_n))$, where X_n is a connected finite CW-complex. We assume the inductive system is unital. The connectedness implies that any projection in A_n has constant rank.

I.6.1.16. The constant functions in A_n form a unital C*-subalgebra isomorphic to $\mathbb{M}_{k_n}$. The restriction of $\phi_{n,n+1}$ to this subalgebra is a unital homomorphism from $\mathbb{M}_{k_n}$ to A_{n+1}. The standard matrix units give a set of k_n rank-one projections in $\mathbb{M}_{k_n}$ which are equivalent and add to 1; hence the images under $\phi_{n,n+1}$ are equivalent and add to 1 and thus are all of the same rank k_{n+1}/k_n. In particular, k_n must divide k_{n+1}, and we say the embedding $\phi_{n,n+1}$ has *multiplicity* $m_n = k_{n+1}/k_n$.

I.6.1.17. Fix an n_0 and set $k = k_{n_0}$. Let p be a nonzero projection in $A_{n_0} = M_k(C(X_{n_0}))$ of rank ρ. Then, for any $n > n_0$, the projection $\phi_{n_0,n}(p)$ has constant rank $\frac{\rho k_n}{k}$.

I.6.1.18. Now suppose that
$$\lim_{n\to\infty} \frac{dim(X_n)}{k_n} = 0 \ .$$
Then, for sufficiently large $n > n_0$, the (constant) rank $\frac{\rho k_n}{k}$ of $\phi_{n_0,n}(p)$ will be greater than $dim(X_n)$.

I.6.1.19. Suppose q and r are also projections in A_{n_0} with $p \perp r$, $q \perp r$, $p + r \sim q + r$. We then have that $rank(q) = \rho$ also. For sufficiently large $n > n_0$, we have that $rank(\phi_{n_0,n}(p)) = rank(\phi_{n_0,n}(q)) > dim(X_n)$ and $\phi_{n_0,n}(p) + \phi_{n_0,n}(r) \sim \phi_{n_0,n}(q) + \phi_{n_0,n}(r)$, so $\phi_{n_0,n}(p) \sim \phi_{n_0,n}(q)$ by I.6.1.10.. Hence $A = \lim_{\to}(A_n, \phi_{n,n+1})$ has cancellation of projections. The argument works identically in matrix algebras, so A has cancellation.

I.6.1.20. Similarly, suppose q is a projection in A_{n_0} of rank $\sigma > \rho$. Then for sufficiently large $n > n_0$ we have that
$$rank(\phi_{n_0,n}(q)) - rank(\phi_{n_0,n}(p)) = \frac{k_n}{k}(\sigma - \rho) > dim(X_n)$$
so $\phi_{n_0,n}(p) \precsim \phi_{n_0,n}(q)$ by I.6.1.11.. Thus A has strict comparison.

Note that simplicity of the inductive limit is not needed for these arguments.

I.6.1.21. In the general simple AH case a similar condition and argument works, although the technical details are a little more complicated. We consider an AH system $(A_n, \phi_{n,n+1})$, where
$$A_n = \bigoplus_j A_{n,j} = \bigoplus_j p_{n,j} M_{\nu_{n,j}}(C(X_{n,j}))p_{n,j}$$
(finite direct sum for each n) and the $\phi_{n,n+1}$ are unital. We may assume without loss of generality that the $X_{n,j}$ are connected finite CW-complexes (). Set $k_{n,j} = rank(p_{n,j})$ for all n, j.

I.6.1.22. We can define multiplicities of partial embeddings as before. In the case where all the $p_{n,j}$ are the identity, this is straightforward: for any n and i, j, there is a nonnegative integer m_{ij} such that if p is any projection n $A_{n,i}$, the local rank of $\phi_{n,n+1}(p)$ in $A_{n+1,j}$ is m_{ij} times the rank of p in $A_{n,i}$. This is still true in the case of general $p_{n,j}$, although slightly tricky to prove (). The number $m_{i,j}$ is the *multiplicity of the partial embedding* of $A_{n,i}$ into $A_{n+1,j}$. By composition we can define the multiplicities of the partial embeddings of $A_{n,i}$ into $A_{m,j}$ whenever $n < m$.

The multiplicities of the partial embeddings can be cleanly calculated as
$$m_{ij} = \frac{k_{n+1,j}}{k_{n,i}}\tau(\phi_{n,n+1}(1_{A_{n,i}}))$$
where τ is any trace on $A_{n+1,j}$; connectedness of $X_{n+1,j}$ implies that $\tau(\phi_{n,n+1}(1_{A_{n,i}}))$ is independent of τ and is exactly the normalized rank of $\phi_{n,n+1}(1_{A_{n,i}})$.

I.6.1.23. We will require that for each n_0 there is an $n > n_0$ such that all the multiplicities of partial embeddings of the $A_{n_0,i}$ into each $A_{n,j}$ are strictly positive. This condition is automatic if the inductive limit is simple; in fact, it is much less restrictive than simplicity of the inductive limit (e.g. it is satisfied in the situation of I.6.1.15.). The condition can be more cleanly described as the requirement that for every n_0 there is an $n > n_0$ such that for every nonzero projection p in A_{n_0} and every trace τ on A_n, $\tau(\phi_{n_0,n}(p)) > 0$.

I.6.1.24. Definition. We say the system $(A_n, \phi_{n,n+1})$ (satisfying the condition of I.6.1.23.) has *slow dimension growth* if

$$\liminf_{n\to\infty} \left[\max_j \frac{dim(X_{n,j})}{k_{n,j}} \right] = 0 \ .$$

I.6.1.25. Theorem. Let $(A_n, \phi_{n,n+1})$ be a unital AH inductive system satisfying the condition in I.6.1.23.. If the system has slow dimension growth, then the inductive limit has cancellation and strict comparison.

I.6.1.26. A more sophisticated version of the last part of the argument shows that the inductive limit of any AH system satisfying the condition in I.6.1.23., and with slow dimension growth, has the (SP) property ().

I.6.1.27. Any AH inductive system satisfying the condition of I.6.1.23. in which the dimensions of the base spaces are uniformly bounded has slow dimension growth. In particular, an AF, AI, or A$\mathbb{T}$ inductive system with simple inductive limit has slow dimension growth.

I.6.1.28. Note that the slow dimension growth condition is a property of inductive systems, not directly of AH algebras. But we can say that a (simple unital) AH algebra has slow dimension growth if it has an AH inductive system with slow dimension growth. Thus a simple unital A$\mathbb{T}$ algebra (AF algebra, AI algebra) has slow dimension growth. An AH algebra with slow dimension growth can easily have AH inductive systems which do not have slow dimension growth: it is fairly easy to even find an AH inductive system for the CAR algebra which does not have slow dimension growth.

I.6.1.29. Since the known examples of simple unital AH algebras with base spaces of dimension ≤ 3 exhaust all possibilities of the Elliott invariant for simple AH algebras of slow dimension growth, it was a reasonable conjecture that all such algebras should be classifiable by the Elliott invariant, i.e. are isomorphic to the known classified examples (the conjecture turns out to be true). The other key question then was whether every simple AH algebra has slow dimension growth (this is a far harder problem than simply constructing AH systems which do not have slow dimension growth).

I.6.1.30. There is also a version of the slow dimension growth condition for ASH systems, but it is clumsy and has not proved too useful. It is cleaner to work with the equivalent condition of finite nuclear dimension ().

Approximate Divisibility

A closely related property is approximate divisibility, introduced in [BKR92].

I.6.1.31. This property makes sense for general (unital) C*-algebras (e.g. the irrational rotation algebras were shown to be approximately divisible in [BKR92] before they were known to be AH algebras), but is only interesting (can only occur) in C*-algebras with reasonably many projections. The more refined notion of $\mathcal{Z}$-stability (), which coincides with approximate divisibility for classifiable C*-algebras with sufficiently many projections, is of more general interest.

Density of Invertibles

It turns out that cancellation is closely related to density of invertible elements.

I.6.1.32. Definition. Let A be a unital C*-algebra. Then A has *stable rank one* if the invertible elements are dense in A. We write $sr(A) = 1$ if A has stable rank one.

I.6.1.33. The terminology is used because this is the base case of the theory of (topological) stable rank, developed in [**?**], the first dimension theory for C*-algebras; a similar purely algebraic theory called *Bass stable rank* already existed and was used in cancellation problems in ring theory [**?**]. Topological stable rank and Bass stable rank agree for C*-algebras [**?**], so the term "stable rank" is unambiguous in the C*-algebra setting. The connection with dimension theory is:

I.6.1.34. Proposition. Let X be a compact metrizable space. Then $sr(C(X)) = 1$ if and only if $dim(X) \leq 1$.

See () for a more complete description of stable rank and an extension of this result.

I.6.1.35. Proposition. Let A be a unital C*-algebra of stable rank one. Then any unital C*-algebra stably isomorphic to A also has stable rank one. In particular, any corner in a matrix algebra over A has stable rank one.

C*-algebras of stable rank one have pleasant nonstable K-theory properties [**?**]:

I.6.1.36. Theorem. Let A be a unital C*-algebra with stable rank one. Then

(i) A has cancellation.

(ii) The natural map from $\mathcal{U}(A)/\mathcal{U}(A)_0$ to $K_1(A)$ is an isomorphism.

See () for a proof.

The relevance to classification is the following result from [BDR91], extending [DNNP92]:

I.6.1.37. Theorem. Let A be a simple unital AH algebra with slow dimension growth. Then $sr(A) = 1$.

It follows from general principles (e.g. ()) that if A has slow dimension growth, then $sr(A) \leq 2$; but the fact that it is actually one is a little surprising, and has pleasant consequences. See () for a proof.

The converse of I.6.1.37. is false (I.6.2.7.).

I.6.1.38. It turns out that all classifiable stably finite simple unital C*-algebras have stable rank one. But the infinite ones do not: a nonunitary isometry cannot be a limit of invertibles (), so a C*-algebra of stable rank one must be stably finite. (Nonetheless, the conclusions of I.6.1.36. hold for these algebras [(i) only holds for nonzero projections]; cf. ().)

I.6.2. The Examples of Villadsen

It seemed unlikely, at least to the author of [Bla93], that all simple AH algebras have slow dimension growth, and there was an evident potential construction of a counterexample outlined there. However, that author lacked the technical expertise to carry out the construction. J. Villadsen had the expertise, and gave the examples in [Vil98b] and [Vil99].

The First Construction

I.6.2.1. Villadsen's first example, or class of examples, used a version of the diagonal construction of I.3.5.5. with particularly simple connecting maps, but with base spaces rapidly increasing in dimension. This construction has come to be called (cf. [**?**]) the "Villadsen I" construction.

I.6.2.2. The construction comes by fixing a "seed space" X, which can be any compact metrizable space but is generally taken to be a finite CW-complex. We then set $X_1 = X$ and, for each n, $X_{n+1} = X_n^{r_n}$, the Cartesian product of r_n copies of X_n for some $r_n \in N$. Then $X_{n+1} = X^{s_n}$, where $s_n = \prod_{j=1}^n r_j$. We also fix a sequence (m_n) of natural numbers > 1. Set $k_1 = 1$ and $k_{n+1} = \prod_{j=1}^n m_j$.

I.6.2.3. We then consider inductive systems of the form $(M_{k_n}(C(X_n)), \phi_{n,n+1})$, where the connecting maps $\phi_{n,n+1}$ are diagonal maps () with diagonal terms either point evaluations or of the form $f \mapsto f \circ \pi$, where π is a coordinate projection from $X_n^{r_n}$ to X_n (different coordinate projections are allowed in different diagonal terms). For example, all the earlier constructions modeled on I.3.5.4. are of this form with all $r_n = 1$. If X is a single point, the UHF inductive systems are obtained. Simple inductive limits of systems of this form are called *VI algebras*. VI algebras are, of course, AH algebras.

The construction can be expanded to allow finite direct sums of matrix algebras of different sizes over $C(X_n)$ at the cost of complicated bookkeeping; if this is done a larger class of simple C*-algebras is obtained including all simple A$\mathbb{T}$ algebras and most, if not all, other classifiable simple AH algebras. The extra bookkeeping will not be worth it for our present purposes, however.

I.6.2.4. The allowance of a few point evaluations is harmless and is an easy way to obtain simple inductive limits if the points are chosen densely as in I.3.5.4.. One must take at least one point evaluation at each step to obtain simplicity, but should let only a (relatively) small number of the homomorphisms at each stage be point evaluations to get the right pathological properties.

I.6.2.5. Villadsen specifically took $X = S^2$, and showed that if the r_n, m_n, and maps were chosen correctly, a simple unital AH (VI) algebra is obtained which has *perforation* in its ordered K_0 group: there is an element $x \in K_0(A)$, of infinite order, such that $x \not\geq 0$ but $nx \geq 0$ for some $n > 1$. In fact, such an example was constructed for any $n > 1$. The reason exploited in the construction is that

I.6.2.6. It follows from I.6.1.25. that these examples do not have slow dimension growth. (They have "moderate dimension growth" in the sense that

$$\limsup_{n\to\infty} \frac{dim(X_n)}{k_n} < \infty\ .)$$

The Second Construction

The VI algebras did not exhibit all pathology possible in AH algebras which do not have slow dimension growth, due to the following observation of Villadsen:

I.6.2.7. Theorem. Every VI algebra has stable rank 1.

Thus a different construction was needed to produce a simple unital AH algebra whose invertible elements are not dense (it was by no means generally believed at the time that such examples necessarily even existed!)

I.6.2.8. Villadsen produced such examples in [Vil99].

I.6.2.9. These examples, while troublingly pathological, did not directly disprove the conjecture that the Elliott invariant is complete for separable simple unital nuclear C*-algebras, since their Elliott invariants were not the same (or, in the VII case, at least were not known to be the same – they are very complicated) as the ones of known classifiable algebras. But worse examples were soon to come.

I.6.3. Rørdam's Example

I.6.3.1. In (), M. Rørdam constructed arguably the most pathological known separable simple unital UCT C*-algebra: one that is finite but not stably finite. A matrix algebra over this example also provides a separable simple unital UCT algebra which is properly infinite but not purely infinite. Such an example provides a clean counterexample to the conjecture that the Elliott invariant is a complete isomorphism invariant, since it has the same Elliott invariant as a certain UCT Kirchberg algebra.

I.6.4. The Examples of Toms

While Rørdam's example showed that there are simple UCT algebras outside the extreme classes of stably finite and purely infinite algebras, there was still hope that each of these classes could be completely classified by the Elliott invariant (in fact, the purely infinite case was already a theorem). But A. Toms dashed the hope for the stably finite case (which was already on shaky ground from Villadsen's examples) by constructing simple unital UCT algebras with the same Elliott invariant as UHF algebras, but not UHF. The device he used to prove his examples were not UHF involved a notion of independent interest and, ultimately, of great importance, the Cuntz semigroup.

The Cuntz Semigroup

I.6.4.1. In [Cun78], J. Cuntz constructed a semigroup which eventually (after many years!) became a critical tool in the analysis and classification of C*-algebras (cf. [BPT08], [APT11]). The idea is to build a semigroup similar to the semigroup $V(A)$ of I.3.3.3., but using all (positive) elements of the C*-algebra A (and matrix algebras over A), not just projections. Cuntz actually passed to the Grothendieck group in [Cun78] and worked with that, but in more recent work it has been better to stick with the semigroup.

I.6.4.2. There are many slightly different equivalence relations and partial orders which can be defined on the set of positive elements of a C*-algebra A (or a local C*-algebra such as $M_\infty(A)$); see [Bla06, II.3.4.8]. Cuntz chose the one which seems to be most useful:

I.6.4.3. Definition. Let A be a (local) C*-algebra, $a, b \in A$. Then $b \precsim a$ (in A) if there are sequences (x_n), (y_n) in A with $x_n a y_n \to b$. We say a and b are *Cuntz equivalent* (in A), written $a \approx b$, if $a \precsim b$ and $b \precsim a$.

Caution: the literature is inconsistent about notation for the various equivalence relations and orders. ✋

I.6.4.4. It is easily checked that $\precsim$ is reflexive and transitive, and $\approx$ is an equivalence relation. (Note that these relations are dependent on the containing C*-algebra A; if necessary, we will write $\precsim_A$ and $\approx_A$.) It is trivial or routine to show:

I.6.4.5. Proposition. Let A be a local C*-algebra. Then

(i) If $a \in A$ and $0 \neq \alpha \in \mathbb{C}$, then $a \approx \alpha a \approx a^* \approx a^* a \approx a a^*$.

(ii) If $a, b \in A$, then $ab \precsim a$ and $ba \precsim a$.

(iii) If $a, b \in A$ and $\overline{a^* A a} = \overline{b^* A b}$, then $a \approx b$.

(iv) If $a \in A_+$, then $a \approx a^n$ for any $n \in \mathbb{N}$.

(v) If $a, b \in A_+$, then $b \precsim a$ if and only if there is a sequence (x_n) in A with $x_n^* a x_n \to b$.

I.6.4.6. Since $a \approx a^* a$ for any a, one can just work with positive elements of A, in which case the partial order is as given in I.6.4.5.(v). One should think of $b \precsim a$ as meaning the support of b is "smaller" ("not larger") than the support of a, although not necessarily in the same "location," and similarly for $a \approx b$.

I.6.4.7. To get a feel for Cuntz equivalence, consider the commutative case, $A = C(X)$ for a compact Hausdorff space X. If $f \in C(X)$, define the *cozero set* $coz(f)$ to be $\{x \in X : f(x) \neq 0\}$ (this set is sometimes called the *support* of f, but this term also sometimes means the closure of $coz(f)$ so is ambiguous). Then $f \approx g$ in $C(X)$ if and only if $coz(f) = coz(g)$. (Here we do not need to restrict to positive elements, which are just nonnegative real-valued continuous functions.) Thus the Cuntz equivalence classes in $C(X)$ are parametrized by the open F_σ's in X (just open sets if X is metrizable), and the ordering is by inclusion.

If we consider a matrix algebra over $C(X)$, say $M_n(C(X)) \cong C(X, \mathbb{M}_n)$, things are more complicated. If $f, g \in C(X, \mathbb{M}_n)_+$, then a necessary condition for $f \approx g$ is that f and g have the same rank at every point, but this is not sufficient for all n if X is large enough to have a compact subspace supporting a nontrivial complex vector bundle (e.g. if X is a finite CW complex of dimension ≥ 3, which contains a 2-sphere). The Cuntz equivalence classes in matrix algebras over $C(X)$ and their ordering can be explicitly described if $dim(X) \leq 3$ (), but become ever more complicated for larger dimensions. For a fuller discussion, see ().

I.6.4.8. If p and q are projections, it is easy to check () that $p \precsim q$ in the sense of I.6.4.3. if and only if $p \precsim q$ in the sense of I.1.6.24., so the notation is consistent. If $p \sim q$, then $p \approx q$; but $p \approx q$ does not imply $p \sim q$ in general (e.g. in O_n, $n \geq 3$) unless A is stably finite ().

I.6.4.9. If A is a C*-algebra, let $W(A)$ be the set of Cuntz equivalence classes of (positive) elements in $M_\infty(A)$. There is a well-defined binary operation on $W(A)$ defined by $[a] + [b] = [diag(a, b)]$, which makes $W(A)$ into an abelian semigroup with identity $[0]$. There is also a well-defined partial order on $W(A)$ defined by $[b] \leq [a]$ if $b \precsim a$, which is compatible with the semigroup operation. There is a natural "inclusion" homomorphism from $V(A)$ to $W(A)$, which is injective if A is stably finite.

I.6.4.10. Definition. If A is a C*-algebra, $W(A)$ with this addition and partial order is the *Cuntz semigroup* of A.

I.6.4.11. The Cuntz semigroup is of primary importance in the stably finite case. If A is stably finite, an element of A is Cuntz equivalent to a projection if and only if it is well supported in the sense of [Bla06, II.3.2.8]. In the stably finite case, $W(A)$ breaks into two disjoint subsets: write $V(A)$ for the subset of $W(A)$ consisting of equivalence classes of projections (or well-supported elements), and $W_0(A)$ (called $W(A)_+$ in some references) the set of equivalence classes of non-well-supported elements. (The notation is consistent with I.1.6.24. in the stably finite case.) Both $V(A)$ and $W_0(A)$ are subsemigroups of $W(A)$ if A is simple (or has stable rank 1); if $x \in V(A)$ and $y \in W_0(A)$, then $x + y \in W_0(A)$. Both $V(A)$ and $W_0(A)$ are cofinal in $W(A)$ in an order-theoretic sense if A is simple, stably finite, and infinite-dimensional.

I.6.4.12. Example. If A is purely infinite (by our definition, i.e. simple and unital), then $W(A)$ consists precisely of two elements: $[0]$ and $[1]$, where $[0]$ is the class of 0 and $[1]$ is the class of 1_A: every nonzero element of $M_\infty(A)$, or even $A \otimes \mathbb{K}$, is Cuntz equivalent to 1_A. These are ordered $[0] < [1]$, and the addition is $x + y = \max(x, y)$. We have $W_0(A) = \{[1]\}$ and $\mathrm{Cu}(A) = W(A)$. The class $[1]$ can be regarded as "∞." In this case, the image of $V(A)$ in $W(A)$ is all of $W(A)$, but the map is not injective in general.

If A is not simple, then $W(A)$ includes the ideal structure of A, so can be complicated. $W(A)$ can even be very complicated for simple A.

I.6.4.13. Example. Let A be a UHF algebra with $K_0(A)$ identified with a subgroup G of $\mathbb{Q}$ with $[1_A] = 1$ as usual. Then $V(A) \cong G_+ = G \cap [0, +\infty)$. Thus for each $r \in G_+$ we have an $x_r \in V(A) \subseteq W(A)$ which is the equivalence class of a projection of trace r in a matrix algebra over A. For each $r \in (0, +\infty)$ we construct an element y_r of $W_0(A)$ as follows. If $n \geq r$, there is a sequence of mutually orthogonal nonzero projections (p_k) in $M_n(A)$ with

$$\sum_{k=1}^{\infty} \tau(p_k) = r \ .$$

Set

$$a_r = \sum_{k=1}^{\infty} 2^{-k} p_k \in M_n(A)$$

and let $y_r = [a_r]$.

It can be shown that every non-well-supported element of $M_\infty(A)$ is Cuntz equivalent to a_r for some unique r: in fact, if b is a non-well-supported positive element of A of norm one, then $b \approx a_r$, where

$$r = \sup_{n \in \mathbb{N}} \tau(b^{1/n})$$

(the fact that A has real rank zero, hence (HP), makes this easy) and thus $W_0(A) = \{y_r : r \in (0, +\infty)\}$. Note that if b is a projection (or well-supported positive element of norm one), then $[b] = x_r$, where r is given by the same formula; in fact, this formula defines the unique dimension function () on A.

It is easy to see that $y_r + y_s = y_{r+s}$ for any $r, s \in (0, +\infty)$, and thus $W_0(A)$ is isomorphic to the additive semigroup $(0, +\infty)$. If $r \in G_+$, $s \in (0, \infty)$, then $x_r + y_s = y_{r+s}$. We have $x_r \leq x_s$ if and only if $r \leq s$ in G_+, and $y_r \leq y_s$ if and only if $r \leq s$ in $(0, +\infty)$. If $r \in G_+$, $s \in (0, +\infty)$, then $y_s \leq x_r$ if and only if $s \leq r$ and $x_r \leq y_s$ if and only if $r < s$. Thus $W(A)$ is isomorphic to $[0, +\infty)$ with the positive points of G "doubled," with the ordering a "lexicographic" order. Note that this semigroup does not have cancellation: we have $x_r + y_s = y_r + y_s$ for any $0 < r \in G_+$, $s \in (0, +\infty)$, but $x_r \neq y_r$.

I.6.4.14. This is one of the relatively few examples where the Cuntz semigroup can be explicitly calculated; it can be explicitly described (as a "monotone completion" of $V(A)$) more generally for C*-algebras of real rank zero, but not far beyond that, although remarkably it can be explicitly described in a similar way for classifiable simple C*-algebras (and only for the classifiable ones!), cf. ().

I.6.4.15. We evidently have $W(M_n(A)) \cong W(A)$ for any A and n. More generally, if A and B are unital C*-algebras which are stably isomorphic (Morita equivalent), then $W(A) \cong W(B)$. However, if A is a C*-algebra, then $W(A \otimes \mathbb{K})$ is usually not isomorphic to $W(A)$; it is $W(A)$ with some "infinite" elements added in general. Thus the Cuntz semigroup is more sensitive to stabilization than K-theory (and does not behave as nicely for inductive limits). There is a "completed" version of $W(A)$, denoted Cu(A), which does behave well under inductive limits and agrees with $W(A \otimes \mathbb{K})$ in good cases. See [CEI08] or [APT11].

I.6.4.16. The Cuntz semigroup is functorial: since any *-homomorphism of C*-algebras is continuous, if $\phi : A \to B$ is a *-homomorphism and $x, y \in A$ with $y \precsim_A x$, then $\phi(y) \precsim_B \phi(x)$.

I.6.4.17. We can form the Grothendieck group of the semigroup $W(A)$, called $K_0^*(A)$ in [Cun78]. $K_0^*(A)$ has a natural ordering induced from $W(A)$ making it an ordered group (in the stably finite case); in fact, the positive cone is exactly the image of $W(A)$ in good cases (not always!). If A is unital, $(K_0^*(A), K_0^*(A)_+)$ has a natural order unit [1], and $(K_0^*(A), K_0^*(A)_+, [1])$ is a scaled ordered group (usually not countable, even if A is separable). For example, if A is any UHF algebra, then $K_0^*(A)$ is algebraically and order-isomorphic to the additive group $\mathbb{R}$. When forming $K_0^*(A)$, it is crucial to work only with $W(A)$ and not the completion with infinite elements (which would force the Grothendieck group to be degenerate).

Toms's Examples

I.6.4.18. The examples constructed by Toms are simple unital AH algebras with unique trace in which $K_0(A)$ is unperforated and isomorphic to the dimension group of a UHF algebra, but $K_0^*(A)$ is perforated. Specifically, he used the notion of radius of comparison (), and showed that an arbitrary radius of comparison can be realized.

Other Uses of the Cuntz Semigroup

I.6.4.19. The original use of the Cuntz semigroup of a C*-algebra A (or, rather, its Grothendieck group $K_0^*(A)$) in [Cun78] was to describe dimension functions on A. This has important bearings on the trace space of A and its relation to the ordered group $K_0(A)$. These matters are described in more detail in III.3..

I.6.4.20. It has often been considered that the Cuntz semigroup should be added to the Elliott invariant. Before the classification theorem, and in light of the examples of Villadsen and Toms (and Rørdam), it was sometimes thought that a broadly applicable classification result might not be possible without it. Fortunately, this turned out not to be the case, although the possibility remains that a considerably broader classification theorem might be possible if the Cuntz semigroup is included in the invariant (cf. ()). But the drawback is that the Cuntz semigroup is very complicated and extremely difficult to calculate in all but the simplest examples, so an invariant including the Cuntz semigroup would be of little practical value.

I.6.5. The Jiang-Su Algebra

There is another crucial example which came about the same time () of a very different nature: an infinite-dimensional simple unital ASH algebra (with slow dimension growth) with the same Elliott invariant as $\mathbb{C}$. This example is technically a counterexample to the conjecture that the Elliott invariant is a complete isomorphism invariant for separable simple unital nuclear C*-algebras (the simple fix is just to add "infinite-dimensional" to the statement of the conjecture, or "non-elementary" for the nonunital case); but far from a pathological counterexample, this algebra has turned out to be a critical tool in the latter parts of the classification project, and it is now regarded as one of the most important C*-algebras of any kind.

An Exact Example

I.6.5.1. We begin with a folklore example of an exact but nonnuclear simple unital C*-algebra with the same Elliott invariant as $\mathbb{C}$. The author noted this example soon after [PV82] appeared, and probably was not the only one. The (arguable) main result of [PV82] (part (i) was previously known) was:

I.6.5.2. Theorem. Let $\mathbb{F}_n$ be the free group on n generators ($2 \leq n \leq \infty$). Then

(i) $C_r^*(\mathbb{F}_n)$ is simple, unital, and exact with unique trace, but not nuclear.

(ii) $C_r^*(\mathbb{F}_n)$ is projectionless and $K_0(C_r^*(\mathbb{F}_n)) \cong \mathbb{Z}$, generated by [1].

(iii) Let $\{g_1, g_2, \dots\}$ be a set of free generators for $\mathbb{F}_n$, and u_k the unitary in $C_r^*(\mathbb{F}_n)$ corresponding to g_k. Then $K_1(C_r^*(\mathbb{F}_n)) \cong \mathbb{Z}^n$, with free abelian generators $\{[u_1], [u_2], \dots\}$. In other words, the map $\mathbb{F}_n \to K_1(C_r^*(\mathbb{F}_n))$ given by $g_k \mapsto [u_k]$ is surjective, and the kernel is precisely the commutator subgroup of $\mathbb{F}_n$.

(Parts (ii) and (iii) are valid also for $n = 1$: $\mathbb{F}_1 = \mathbb{Z}$, so $C_r^*(\mathbb{F}_1) = C^*(\mathbb{F}_1) \cong C(\mathbb{T})$.)

I.6.5.3. Since any subgroup of a free group is free, the commutator subgroup of $\mathbb{F}_\infty$ is isomorphic to $\mathbb{F}_\infty$. Let γ be an embedding of $\mathbb{F}_\infty$ into $\mathbb{F}_\infty$ mapping $\mathbb{F}_\infty$ onto the commutator subgroup. Let G be the direct limit of the stationary system

$$\mathbb{F}_\infty \xrightarrow{\gamma} \mathbb{F}_\infty \xrightarrow{\gamma} \mathbb{F}_\infty \xrightarrow{\gamma} \cdots$$

Then G is a countable group equal to its own commutator subgroup. If ϕ is the embedding of $C_r^*(\mathbb{F}_\infty)$ into $C_r^*(\mathbb{F}_\infty)$ induced by γ, then $C_r^*(G)$ is isomorphic to the inductive limit of the stationary system

$$C_r^*(\mathbb{F}_\infty) \xrightarrow{\phi} C_r^*(\mathbb{F}_\infty) \xrightarrow{\phi} C_r^*(\mathbb{F}_\infty) \xrightarrow{\phi} \cdots$$

so $C_r^*(G)$ is separable, simple, unital, projectionless, and exact with unique trace, but not nuclear since G is not amenable (). The map ϕ induces the identity map on K_0 and the zero map on K_1, so $K_0(C_r^*(G)) \cong \mathbb{Z}$ and $K_1(C_r^*(G)) = 0$. (It is unclear whether G or $C_r^*(G)$ depends up to isomorphism on the choice of γ.)

Strongly Self-Absorbing C*-Algebras

The Jiang-Su algebra $\mathcal{Z}$ has some remarkable tensor product properties. To begin with, $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$. But much more is true. We first make some definitions:

I.6.5.4. Definition. Let A and B be separable C*-algebras, with B unital, and ϕ, ψ homomorphisms from A to B. Then ϕ and ψ are *approximately unitarily equivalent* if there is a sequence (u_n) of unitaries in B such that $u_n^*\phi(x)u_n \to \psi(x)$ for every $x \in A$.

It is easily checked that this is indeed an equivalence relation on homomorphisms. Approximately unitarily equivalent homomorphisms induce the same homomorphisms on K-theory.

I.6.5.5. Definition. A separable unital C*-algebra $D \ncong \mathbb{C}$ is *strongly self-absorbing (SSA)* if the homomorphism $x \mapsto x \otimes 1$ from D to $D \otimes D$ is approximately unitarily equivalent to an isomorphism.

This notion was introduced in [TW07], but the closely related property of approximately inner flip was considered in [ER78], and the factor version was already critical in [Con76].

I.6.5.6. A SSA C*-algebra is infinite-dimensional. Here are some basic properties of a SSA C*-algebra D ():

(i) D has *approximately inner flip*, i.e. the flip automorphism σ of $D \otimes D$ defined by $\sigma(x \otimes y) = y \otimes x$ is approximately unitarily equivalent to the identity map. Thus by [ER78], D must be nuclear.

(ii) D must be simple.

(iii) If D is in the UCT class, then $K_1(D) = 0$ and $K_0(D)$ is torsion-free and rank ≤ 1, by the Künneth Theorem for Tensor Products ().

(iv) If D has a trace, the trace is unique. If D has no trace, it is purely infinite.

(v) $D \cong D^{\otimes\infty}$, and in fact the map $x \mapsto x \otimes 1$ from D to $D \otimes D^{\otimes\infty} \cong D^{\otimes\infty}$ is approximately unitarily equivalent to an isomorphism.

I.6.5.7. EXAMPLES. (i) Some UHF algebras, including the CAR algebra, are SSA. In fact, $\mathbb{M}_{\boldsymbol{m}}$, $\boldsymbol{m} = 2^{e_2}3^{e_3}5^{e_5}\cdots$, is SSA if and only if each e_p is either 0 or ∞ ($\mathbb{M}_{\boldsymbol{m}}$ is of *infinite type*).

(ii) By (), O_2 and O_∞ are SSA.

(iii) For a separable unital C*-algebra A to be SSA, it is not sufficient for A to be isomorphic to $A \otimes A$ or even to $A^{\otimes\infty}$: the commutative C*-algebra $A = C([0,1]^\infty)$ of continuous functions on the Hilbert cube satisfies $A \cong A^{\otimes\infty}$, but is not SSA. For A to be SSA, it is also not sufficient that A have approximately inner flip: $\mathbb{M}_n$ and $M_n(O_\infty)$ for any n, and any UHF algebra, have approximately inner flip. (It may be that if A is unital and has approximately inner flip, then $A^{\otimes\infty}$ is SSA, or even that A is SSA if and only if A is unital, $A \cong A \otimes A$, A has approximately inner flip, and $A \not\cong \mathbb{C}$. Note that there are nonunital C*-algebras satisfying the last three conditions, including $\mathbb{K}$ and probably the A of (), and stabilizations of SSA algebras.)

I.6.5.8. THEOREM. The Jiang-Su algebra $\mathcal{Z}$ is strongly self-absorbing.

I.6.5.9. In fact, O_2, O_∞, $\mathcal{Z}$, the UHF algebras of infinite type (I.6.5.7.(i)), and the tensor products of such UHF algebras with O_∞ form a complete list of SSA C*-algebras satisfying the UCT (cf. [Win11]). Thus, if the UCT holds, this is a complete list of all SSA C*-algebras. A finite or infinite tensor product of SSA C*-algebras is SSA, but it is known that this list is closed under such tensor products. All the algebras in the list except $\mathcal{Z}$ have real rank zero, but $\mathcal{Z}$ is projectionless.

$\mathcal{Z}$-Stability

I.6.5.10. If A is a simple unital C*-algebra with weakly unperforated K_0, then it is intuitively clear and is not difficult to show that $A \otimes \mathcal{Z}$ is a simple unital C*-algebra with the same Elliott invariant as A. Thus the Elliott invariant cannot distinguish between A and $A \otimes \mathcal{Z}$, i.e. in any class $\mathcal{C}$ of simple C*-algebras classified by the Elliott invariant, for every A for which both A and $A \otimes \mathcal{Z}$ are in $\mathcal{C}$, we must have $A \cong A \otimes \mathcal{Z}$. Note that since $\mathcal{Z}$ is ASH, if A is ASH so is $A \otimes \mathcal{Z}$.

I.6.5.11. DEFINITION. A C*-algebra A is *$\mathcal{Z}$-stable* if $A \cong A \otimes \mathcal{Z}$.

I.6.5.12. More generally, if D is a C*-algebra with $D \cong D \otimes D$, we may say that a C*-algebra A is *D-stable* if $A \cong A \otimes D$. Since $\mathbb{K} \cong \mathbb{K} \otimes \mathbb{K}$, we may define $\mathbb{K}$-stability: A is $\mathbb{K}$-stable if and only if it is stable in the sense of (). The notion of D-stability is especially interesting if D is SSA. O_2-stability and O_∞-stability are important in KIRCHBERG's work on purely infinite C*-algebras (). If D is the universal UHF algebra, D-stability has been examined in [Bla90a], [Rør91], [Rør92a], and [Win14].

I.6.5.13. It stands to reason (cf. I.6.5.10.) that classifiable C*-algebras should be $\mathcal{Z}$-stable, and this turns out to be correct (): a simple nonelementary C*-algebra in the UCT class with finite nuclear dimension is $\mathcal{Z}$-stable (and conversely a $\mathcal{Z}$-stable simple nuclear C*-algebra in the UCT class has finite nuclear dimension).

Note that if A is $\mathcal{Z}$-stable, so is $A \otimes B$ for any C*-algebra B. In particular, since $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, $\mathcal{Z}$ itself is $\mathcal{Z}$-stable, as is $A \otimes \mathcal{Z}$ for any A.

I.6.5.14. The algebra $\mathcal{Z}$ is the smallest C*-algebra (in the unital category) which can reasonably be used for stability: all the other strongly self-absorbing (UCT) algebras absorb $\mathcal{Z}$, i.e. are $\mathcal{Z}$-stable. It is thus the best C*-analog of the hyperfinite II_1 factor $\mathcal{R}$. Even in the general (not necessarily unital) category, $\mathcal{Z}$ and $\mathbb{K}$ are the two appropriate C*-algebras for stability (in two different senses), although the C*-algebra $\mathcal{W}$ () might have the same effect as $\mathcal{Z}$ on stably projectionless algebras.

I.6.6. Status at the Millennium

Let us take stock of the situation as of 2000 (maybe 1998 would be a better break point).

I.6.6.1. What we knew:

(i) Except for the UCT problem, the purely infinite (separable nuclear) case was finished: The Elliott invariant, which simplifies in this case to simply the K-theory and the position of the unit in K_0, is a complete isomorphism invariant, and all possible values of the invariant (limited only by the elementary restriction of countability) are realized.

(ii) The stably finite case was in a certain sense finished too: ASH examples with dimension ≤ 2 of all possible values of the Elliott invariant in the weakly unperforated case were explicitly known, and the invariant was known to be complete for ASH algebras of suitably slow dimension growth. In other words, we had a "classifiable class" of simple C*-algebras and a complete classification of them by an invariant which was computable and whose possible values could be cleanly described.

(iii) There was, on the other hand, a considerable zoo of bizarre separable simple unital nuclear C*-algebras (with more and worse soon to come) which appeared to defy any reasonable classification.

I.6.6.2. What we didn't know:

(iv) How can we tell whether a given C*-algebra is in the classifiable class? In the stably finite case, the hypothesis that the algebra could be written as an inductive limit in a certain way was an extremely undesirable one which was very difficult if not impossible to check in many cases, e.g. in the case of a crossed product of a compact metric space by a minimal homeomorphism (it was hard enough to verify in the very special case of an irrational rotation algebra). This hypothesis could even be regarded as begging the question: it was easily conceivable (and was even still thought likely by some specialists) that the class of ASH algebras was only a tiny subclass of the stably finite separable simple nuclear C*-algebras, so special that the techniques from linear algebra and ordinary finite-dimensional topology used to classify them worked there and only there. The fact that there were simple nuclear C*-algebras, even simple AH algebras, which fell outside the known methods of classification suggested that reasonable and usable necessary and sufficient conditions for classifiability might not be easy to obtain, perhaps even in the restricted class of AH algebras.

I suppose we should add:

(v) The UCT problem: is every separable simple nuclear C*-algebra in the bootstrap class?

The UCT problem seems a little different from the problem in (iv) (maybe just because we have been able to understand and answer the problem in (iv)!) in that there are very broad classes of nuclear C*-algebras known to be in the bootstrap class, and in particular any (separable) nuclear C*-algebra arising from virtually any known construction will satisfy the UCT, so the UCT loose end may not have as much practical effect as the other one (which was certainly more than a "loose end"!)

I.6.6.3. Almost all work in the classification program in the years 2000-2015 was concerned at least indirectly with problem (iv). (Actually, by 2000 the first work to circumvent an ASH hypothesis had already begun with LIN'S notion of tracial rank, an outgrowth of POPA'S technique of approximation in corners.) Much of this work is rather technical to describe, and the "right" approach was only gradually found, but the end results were ultimately very satisfying.

I.6.7. The Nonunital Case

By 2015, the unital case was finished. A considerable amount of work had begun on the nonunital case (including construction of what turned out to be all examples), which was more challenging for a number of reasons. Beside the mere technical problems of adapting the unital arguments to the nonunital case, which were so substantial that the nonunital case had to wait until alternate approaches were made to parts of the unital results, there are at least two intrinsic problems which are much more than technical:

I.6.7.1. The most serious problem is that there is no canonical normalization of traces on a (simple) nonunital C*-algebra; indeed, there may be both bounded and unbounded traces. One can at least avoid dealing with unbounded traces by passing to a hereditary C*-subalgebra of the Pedersen ideal (), but even then the trace space need not be compact and is not a stable isomorphism invariant. See () for an illustration. In this case, and more generally if the algebra (stably) contains a nonzero projection, the difficulty can be avoided by cutting down to a corner, which effectively reduces the classification of such algebras to the unital case (cf. ()). But in the stably projectionless case, it is a quite delicate matter to find a stably isomorphic version with compact full trace space, and it is not yet even known to be possible for all projectionless separable simple nuclear C*-algebras (the finite nuclear dimension UCT case is hard enough!)

I.6.7.2. Another serious difficulty is that the K-theory of a (simple) nonunital C*-algebra is not internal: K_0-elements need not have any relation to projections in the algebra, and there are no unitaries to make K_1-elements. One must work indirectly using the unitization. And a projectionless simple C*-algebra can not only have nontrivial K_0 – the pairing of K_0 with the traces (which is even delicate to define properly in the nonunital case, cf. ()) can even be nontrivial.

I.7. Recognizing Classifiability

I.7.1. Regularity

I.7.1.1. One of the main themes behind the classification program as it has developed is that there is a strict dichotomy or "0-1 law": there is a class of separable simple nuclear C*-algebras which are very well behaved and which can be classified by a relatively simple invariant, and a menagerie of other separable simple nuclear C*-algebras which are very badly behaved and defy reasonable description or classification (not to mention a huge unwashed horde of C*-algebras which fail to be separable, simple, and/or nuclear). To paraphrase the nursery rhyme, when a (separable nuclear) simple C*-algebra is good (regular), it is very, very good, and when it is bad (irregular) it is horrid.

I.7.1.2. Regularity comes in many forms:

Finite nuclear dimension

Nuclear dimension ≤ 1

$\mathcal{Z}$-stability

Strict comparability

Well-behaved Cuntz semigroup

all of which are equivalent (there are other actual or potential forms too). Significantly, and fortunately, regularity is automatic in many standard constructions and examples, so the regular simple C*-algebras form a vast subclass including a large fraction of the simple C*-algebras which arise "naturally" in the theory or in applications.

I.7.2. Dimension Theory for C*-Algebras

One clean way to phrase the characterization of classifiability is via an appropriate notion of dimension for C*-algebras. But unlike for compact metrizable spaces, there are many inequivalent ways to define the "dimension" of a (separable) C*-algebra, and the right one for classification was by no means obvious, and in fact was a long time coming, with much trial and error (it was only gradually even realized that classifiability could be characterized by a notion of dimension). And some of the "wrong" ways to define C*-dimension are really not mistakes at all, and contribute insight into the structure of C*-algebras too.

Dimension Theory for Spaces

It is a remarkable and significant fact that all reasonable notions of topological dimension coincide for separable metrizable spaces, and in particular for compact metrizable spaces, and that the topological dimension of $\mathbb{R}^n$ is indeed n. The three standard types of topological dimension are covering dimension and small and large inductive dimension (there are several others also; we will discuss only one of these below). The inductive dimensions do not seem to have reasonable C*-analogs, and have not been used in C*-algebra theory; but covering dimension can be adapted in various important ways.

We will not attempt to give even a brief treatment of topological dimension here. Standard references are the classic [HW41], [Nag65], [Nag70], [AP73], [Pea75], [Eng95], and a more complete summary can be found in [Bla]. We will, however, define the two dimension theories we will use.

I.7.2.3. Definition. Let X be a topological space, and $\mathcal{U} = \{U_i : i \in I\}$ an open cover of X. Then $\mathcal{U}$ has *order* $\leq n$ if, whenever $i_1, \dots, i_{n+2}$ are distinct elements of I, then

$$U_{i_1} \cap \cdots \cap U_{i_{n+2}} = \emptyset$$

(i.e. every point of X is contained in at most $n+1$ of the sets). $\mathcal{U}$ has *order* n if it has order $\leq n$ but not order $\leq n-1$, i.e. there are distinct $i_1, \dots, i_{n+1}$ in I such that

$$U_{i_1} \cap \cdots \cap U_{i_{n+1}} \neq \emptyset \ .$$

I.7.2.4. Definition. Let X be a nonempty topological space. The *covering dimension* $dim(X)$ of X is $\leq n$ if every finite open cover of X has a refinement of order $\leq n$. Say $dim(X) = n$ if $dim(X) \leq n$ and it is false that $dim(X) \leq n-1$. If $dim(X) \leq n$ is false for all n, then $dim(X) = \infty$. Set $dim(\emptyset) = -1$.

I.7.2.5. Most, but not all, authors use $dim(X)$ to denote covering dimension, as we will; the notation $cov(X)$ is also sometimes used. But in some references such as [HW41], $dim(X)$ means something else (small inductive dimension in [HW41]).

There is another notion of topological dimension which seems to be rarely used in topology, but has a natural adaptation to C*-algebras: essential dimension.

I.7.2.6. If X is a topological space of "dimension" less than n, and f is a map from X into $\mathbb{R}^n$ (or the closed unit ball B^n), the range of f should be a set of dimension $< n$ and should thus have dense complement; so it should be possible to perturb f to a uniformly close function g such that the range of g does not include some specified point, e.g. the origin. Actually, the existence of such pathological functions as space-filling curves casts doubt on the intuition behind this argument, but it turns out rather remarkably that the conclusion is correct.

Another argument leading to the conclusion is that a system of n equations in fewer than n variables "usually" does not have a solution. In special cases there might happen to be a solution, but it should be possible to perturb such a system to a "nearby" system with no solution. On the other hand, it is essentially the content of the Inverse Function Theorem that if we have a "smooth" system of n equations in n variables which has a locally unique solution, then any "nearby" smooth system also has a (locally unique) solution.

I.7.2.7. Definition. Let X be a topological space. A map (continuous function) $f : X \to B^n$ is *inessential* if for every $\epsilon > 0$ there is a map $g : X \to B^n$ with $\|f - g\|_\infty = \sup\{\|f(x) - g(x)\| : x \in X\} < \epsilon$ and with $\mathbf{0} \notin g(X)$. A map f from X to B^n is *essential* if it is not inessential, i.e. every map sufficiently uniformly close to f has $\mathbf{0}$ in the range.

I.7.2.8. This use of the term *essential* is somewhat nonstandard. In topology, it is usually said that a map $f : X \to Y$ is essential if it is not homotopic to a constant map. Thus any map from a topological space to a contractible space such as B^n would be inessential. We will use the term in our sense only in relation to maps into B^n.

I.7.2.9. Definition. Let X be a topological space. The *essential dimension* $ed(X)$ of X is the largest n such that there is an essential map from X to B^n. We say $ed(X) = \infty$ if there is an essential map from X to B^n for all n, and $ed(X) = 0$ if every map from X to $B^1 = [-1, 1]$ is inessential. Set $ed(\emptyset) = -1$.

I.7.2.10. It can be shown that $ed(X) = dim(X)$ for any normal space X (and for any completely regular space if covering dimension is defined using covers by cozero sets of real-valued continuous functions, cf. [Ale47]). In particular, essential dimension coincides with all other standard topological dimensions for separable metrizable spaces.

I.7.2.11. Essential dimension can be nicely rephrased in terms of real-valued continuous functions, leading to fairly obvious (in retrospect!) generalizations to C*-algebras. If X is a compact Hausdorff space, denote by $C_{\mathbb{R}}(X)$ the ring of real-valued continuous functions on X. A function in $C_{\mathbb{R}}(X)$ is invertible if and only if it never takes the value 0. An n-tuple $(f_1, \dots, f_n)$ of elements of $C_{\mathbb{R}}(X)$ generates a proper ideal if and only if there is a point of X where all the f_k simultaneously vanish. Such an n-tuple can be regarded as a continuous function from X to $\mathbb{R}^n$ (automatically bounded since X is compact), and the n-tuple generates a proper ideal if and only if this function has $\mathbf{0}$ in its range. Thus $ed(X) \leq n$ if and only if every $(n+1)$-tuple in $C_{\mathbb{R}}(X)$ can be uniformly approximated arbitrarily closely by an $(n + 1)$-tuple which generates $C_{\mathbb{R}}(X)$ as an ideal, i.e. as a (left) $C_{\mathbb{R}}(X)$-module.

Stable Rank

I.7.2.12. M. Rieffel [Rie83b] used the motivation of I.7.2.11. to define the first dimension theory for C*-algebras, (topological) stable rank. Actually, he had a second motivation too: a theory called *Bass stable rank* (originally *stable range*) had already been developed and applied to cancellation problems in ring theory [Bas64]. Rieffel noticed the similarity of Bass stable rank to essential dimension as described in I.7.2.11., and made the following definition:

I.7.2.13. Definition. Let A be a unital C*-algebra (or, more generally, a unital Banach algebra). Let $lg_n(A)$ be the set of n-tuples of elements of A which generate A as a left A-module. The *(topological) stable rank* $sr(A)$ is the smallest n such that $lg_n(A)$ is dense in A^n (with respect to the obvious topology, the product of the norm topology). If there is no such n, set $sr(A) = \infty$.

This should properly be called the *left topological stable rank* of A; right topological stable rank is defined analogously. Because of the involution, the left and right topological stable ranks of a C*-algebra coincide. Rieffel used the name "topological stable rank" to distinguish it from Bass stable rank, a purely algebraic notion; but it was shown in [HV84] that the topological stable rank of a C*-algebra coincides with the Bass stable rank (Rieffel established one inequality), so the term "stable rank" is unambiguous for C*-algebras.

I.7.2.14. Because $C(X)$ consists of complex-valued continuous functions, topologically functions from X to $\mathbb{R}^2$, the stable rank of $C(X)$ is roughly half the dimension of X. Because Rieffel used the indexing convention of Bass stable rank, which was appropriate for the ring-theoretic cancellation problems but differed by one from the standard indexing of topological dimension, the exact relationship is

$$sr(C(X)) = \left\lfloor \frac{ed(X)}{2} \right\rfloor + 1 = \left\lfloor \frac{dim(X)}{2} \right\rfloor + 1$$

where $\lfloor\cdot\rfloor$ denotes "integer part of" (with $\lfloor\infty\rfloor = \infty + 1 = \infty$). Thus for example $sr(C(X)) = 1$ if and only if $dim(X)$ is 0 or 1.

I.7.2.15. Note that 1 is the minimum value possible for $sr(A)$ for any A ($sr(A)$ is of course a natural number or ∞). We have $sr(A) = 1$ if and only if the left invertible elements of A are dense in A. It follows easily from I.1.1.2.(vii) that if $sr(A) = 1$, then every left (or right) invertible element of A is invertible [if x is left invertible and $yx = 1$, approximate y by a left invertible z closely enough that zx is invertible. Thus z is right invertible, hence invertible, so $x = z^{-1}(zx)$ is invertible], and in particular $sr(A) = 1$ if and only if the invertible elements of A are dense in A ().

The following observations are almost trivial:

I.7.2.16. Proposition. (i) If A is a unital C*-algebra and J a closed ideal of A, then $sr(A/J) \leq sr(A)$.
(ii) If $(A_n, \phi_{n,n+1})$ is an inductive system of unital C*-algebras, and $A = \lim_{\to}(A_n, \phi_{n,n+1})$, then

$$sr(A) \leq \liminf sr(A_n) \ .$$

We can have strict inequality in either (i) or (ii), as easy commutative examples show.

I.7.2.17. The behavior of stable rank under taking matrix algebras is slightly unexpected: $sr(M_n(A))$ is roughly $\frac{sr(A)}{n}$. Because of the indexing convention and the fact that stable rank must be an integer, the precise relationship is:

I.7.2.18. Theorem. Let A be a unital C*-algebra, and $n \in \mathbb{N}$. Then

$$sr(M_n(A)) = \left\lceil \frac{sr(A) - 1}{n} \right\rceil + 1$$

where $\lceil\cdot\rceil$ denotes "least integer $\geq$" (with $\lceil\infty\rceil = \infty - 1 = \infty$).

Thus, in particular, $sr(M_n(A)) = 1$ if and only if $sr(A) = 1$, and if $1 < sr(A) < \infty$, then $sr(M_n(A)) = 2$ for sufficiently large n.

I.7.2.19. This theorem is motivated by an alternate interpretation of stable rank in terms of left invertibility of nonsquare matrices: an element of A^n, regarded as an $n \times 1$ matrix, is in $lg_n(A)$ if and only if it is left invertible. Thus $sr(A)$ is the smallest n such that the set of left invertible $n \times 1$ matrices over A are dense in all $n \times 1$ matrices. Rieffel proved (adapting a result from Bass stable rank) that the left invertible $(m+1) \times 1$ matrices over a unital C*-algebra A are dense in the $(m+1) \times 1$ matrices if and only if the left invertible $(m+k) \times k$ matrices over A are dense in the $(m+k) \times k$ matrices for some (hence all) k. For fixed m, as $k \to \infty$ the $(m+k) \times k$ matrices become more and more square (but never quite get there).

I.7.2.20. As a result of the matrix behavior, it is questionable whether stable rank should be truly regarded as a "dimension theory" for C*-algebras. It has nonetheless been very useful in dealing with cancellation problems. Stable rank 1 has been particularly important, in classification and elsewhere.

I.7.2.21. If A is infinite, it follows immediately from I.7.2.15. that $sr(A) > 1$. In fact, RIEFFEL showed by a simple argument that $sr(A) = \infty$ if A is properly infinite. (The stable rank of the Toeplitz algebra is 2.)

One of the most important and useful results of [Rie83b] is a far-reaching noncommutative analog of the fact that $dim(X \times \mathbb{T}) \leq dim(X) + 1$ for any X:

I.7.2.22. THEOREM. Let A be a unital C*-algebra, and $\alpha \in Aut(A)$. Then

$$sr(A \rtimes_\alpha \mathbb{Z}) \leq sr(A) + 1 \ .$$

I.7.2.23. It followed that the stable rank of the irrational rotation algebras is at most 2 (it is now known to actually be 1). RIEFFEL'S main immediate goal in developing the theory of stable rank was to obtain cancellation results for the irrational rotation algebras, which this fact accomplished.

I.7.2.24. Stable rank extends to nonunital C*-algebras by defining the stable rank of A to be the stable rank of its unitization. This extended notion has most of the same properties as before. But the stable rank of any stable C*-algebra, even a "purely infinite" one like $O_2 \otimes \mathbb{K}$, is either 1 or 2 ($sr(O_2 \otimes \mathbb{K}) = 2$).

Real Rank

In 1991, L. BROWN and PEDERSEN introduced a variation of stable rank which is potentially more similar to topological dimension [BP91]. Real rank has so far not proved to be as generally useful as stable rank, and less is known about its general properties, but the base case of real rank zero is at least as important as the base case of stable rank 1.

I.7.2.25. DEFINITION. Let A be a unital C*-algebra. If $n \geq 0$, then A has *real rank* $\leq n$, written $rr(A) \leq n$, if $A_{sa}^{n+1} \cap lg_{n+1}(A)$ is dense in A_{sa}^{n+1}, where A_{sa} is the set of self-adjoint elements of A. Say A has real rank n if $rr(A) \leq n$ and it is false that $rr(A) \leq n-1$. If $A_{sa}^{n+1} \cap lg_{n+1}(A)$ is never dense in A_{sa}^{n+1} for any n, set $rr(A) = \infty$.

I.7.2.26. Note the different indexing convention for real rank; we can have $rr(A) = 0$ (see the discussion below). The indexing convention was designed to make real rank the same as dimension for unital commutative C*-algebras: if X is a compact Hausdorff space, then $C(X)_{sa} = C_{\mathbb{R}}(X)$, so $rr(C(X)) = ed(X) = dim(X)$.

I.7.2.27. Real rank appears to behave similarly to stable rank with regard to matrices. We should have

$$rr(M_n(A)) = \left\lceil \frac{rr(A)}{2n-1} \right\rceil$$

for any A and n; this is not known in general, but is true at least for commutative A [BE91].

I.7.2.28. One might think that $rr(A)$ is roughly $2 \cdot sr(A)$ for any A, adjusted by 1 because of the indexing conventions, but this fails dramatically in general. In fact, if A is purely infinite, then $sr(A) = \infty$ (I.7.2.21.) but $rr(A) = 0$ ([Zha90]; cf. [Bla06, V.3.2.12]). Thus real rank is an invariant of independent interest from stable rank, and gives different information about the algebra. Stable rank and real rank are not entirely unconnected, however: it is always true that $rr(A) \leq 2 \cdot sr(A) - 1$, and if $rr(A) = 0$, then $sr(A) = 1$ if and only if A has cancellation ([BH82]; cf. [Bla06, V.3.2.15]).

I.7.2.29. The base case $rr(A) = 0$ is especially important and well understood. A unital C*-algebra A has real rank zero if and only if the invertible self-adjoint elements of A are dense in A_{sa}. To see what this means, if $rr(A) = 0$ and $x = x^* \in A$ with, say, $\sigma(x) = [-1, 1]$, then x can be approximated arbitrarily closely by an element $y = y^*$ such that the spectrum of y has a gap around 0. By translating and repeatedly approximating, x can be approximated arbitrarily by a self-adjoint element z whose spectrum is contained in a union of very short line segments separated by gaps. Then by functional calculus z, and hence x, can be closely approximated by a self-adjoint element with finite spectrum (a finite linear combination of mutually orthogonal projections). Conversely, if every self-adjoint element in A can be approximated by an element of finite spectrum (finite linear combination of mutually orthogonal projections), the approximating element can be slightly perturbed to be invertible by just changing a zero coefficient to a small nonzero one. Thus A has real rank zero if and only if it has (FS) (I.3.4.1.). (Real rank 0 is also equivalent to (HP) (I.3.4.7.), a trickier result to prove, cf. III.1.1.5..) Thus real rank zero is a strong existence of projections condition.

I.7.2.30. Real rank, like stable rank, can be extended to nonunital C*-algebras by unitization. Most properties, e.g. the results of I.7.2.29., persist in the nonunital case.

Tracial Rank

Radius of Comparison

Decomposition Rank and Nuclear Dimension

Two additional dimension theories, closely related but subtly different, were introduced more recently: decomposition rank and nuclear dimension. These, particularly nuclear dimension, have proved to be the "right" dimension theories for applications to classification.

These dimension theories are based on the following characterization of covering dimension for spaces. This characterization was implicit in [HW41], but was apparently not explicitly described until [KW04].

I.7.2.31. Definition. Let X be a topological space. An open cover $\mathcal{U}$ of X is *n-decomposable* if $\mathcal{U}$ is a disjoint union $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{n+1}$ of $n+1$ subcollections, each of which consists of pairwise disjoint sets.

The $\mathcal{U}_j$ are often thought of as "colors": an n-decomposable open cover can be colored with $n+1$ colors so that no overlapping sets are colored the same.

I.7.2.32. It is obvious that an n-decomposable open cover has order $\leq n$. The converse is false: consider a cover of a circle by three arcs. (A cover is 0-decomposable if and only if it has order zero, i.e. consists of a partition into clopen subsets.)

However, we do have:

I.7.2.33. Proposition. Let X be a normal topological space, and $0 \leq n < \infty$. Then every open cover of X of order n has an n-decomposable refinement.

Proof: Let $\mathcal{V}$ be an open cover of X of order n. Since $\mathcal{V}$ is locally finite, there is a partition of unity $\{f_i\}$ subordinate to $\mathcal{V}$, which defines a continuous function ϕ from X to $|K|$ for a simplicial complex K of dimension $\leq n$. Let K' be the barycentric subdivision of K. Then each vertex σ of K' is the barycenter of a unique simplex Δ_σ of K; let $d(\sigma) \in \{0, \dots, n\}$ be the dimension of Δ_σ.

Since $|K'|$ can be identified with $|K|$, ϕ may be regarded as a map from X to $|K'|$. For each vertex σ of K', let U_σ be the preimage under ϕ of the star of σ, the points of $|K'|$ whose σ-coordinate is positive. Then $\mathcal{U} = \{U_\sigma\}$ is an open cover of X which refines $\mathcal{V}$. For each k, $0 \leq k \leq n$, let

$$\mathcal{U}_k = \{U_\sigma : d(\sigma) = k\} .$$

Then $\mathcal{U}$ is the disjoint union of the $\mathcal{U}_k$. If σ, τ are vertices of K' with $d(\sigma) = d(\tau) = k$, then σ and τ are the barycenters of distinct k-simplexes in K, so the stars of σ and τ are disjoint, and thus U_σ and U_τ are disjoint. So $\mathcal{U}$ is n-decomposable. 👍

I.7.2.34. Corollary. Let X be a normal topological space, and $n \geq 0$. Then $dim(X) \leq n$ if and only if every finite open cover of X has an n-decomposable refinement.

I.7.2.35. There is one property of topological dimension which fails dramatically for nuclear dimension: behavior under products. If X and Y are topological spaces and $X \times Y$ is normal (so X and Y are also normal), then

$$\max(dim(X), dim(Y)) \leq dim(X \times Y) \leq dim(X) + dim(Y)$$

(the first inequality holds since $X \times Y$ has closed subspaces homeomorphic to X and Y). The second inequality is "usually" an equality, although there are pathological examples where it is strict, and thus the first inequality is "usually" strict. So if X and Y are locally compact and second countable, then

$$\max(dim_{nuc}(C_0(X)), dim_{nuc}(C_0(Y))) \leq dim_{nuc}(C_0(X) \otimes C_0(Y)) \leq dim_{nuc}(C_0(X)) + dim_{nuc}(C_0(Y)) .$$

We could ask in general for (separable) C*-algebras A and B, do we have

$$\max(dim_{nuc}(A), dim_{nuc}(B)) \leq dim_{nuc}(A \otimes B) \leq dim_{nuc}(A) + dim_{nuc}(B) \text{ ?}$$

The second inequality may hold in general, although is of limited interest since "most" C*-algebras have infinite nuclear dimension. But the first inequality can fail badly: if A is a separable simple UCT C*-algebra with $dim_{nuc}(A) = \infty$ (i.e. not classifiable), we have $dim_{nuc}(A \otimes \mathcal{Z}) \leq 1$ (we can even have $dim_{nuc}(A \otimes \mathcal{Z}) = 0$ – if A is Toms's example (), then $A \otimes \mathcal{Z}$ is a UHF algebra). The first inequality might not be expected anyway, since unlike in the commutative case A and B are usually not quotients of $A \otimes B$.

I.7.3. Quasidiagonality

Quasidiagonality, described in [**?**] as "external approximation by finite-dimensional algebras," is a somewhat mysterious property of some C*-algebras. Although it is now understood much better than it was a few years ago, we are still far from a complete understanding.

In this section, for ease of exposition we will restrict to separable C*-algebras and representations on separable Hilbert spaces, although the theory and most results can be adapted to the nonseparable case in a straightforward way.

Quasidiagonality was introduced in (), and some fundamental properties were obtained by Voiculescu in [**?**].

I.7.3.1. Definition. Let $\mathcal{H}$ be a separable Hilbert space. A concrete C*-algebra B of operators on $\mathcal{H}$ is a *quasidiagonal algebra of operators* if there is an increasing sequence (p_n) of finite-rank projections in $\mathcal{B}(\mathcal{H})$ with $\sup p_n = 1$, such that

$$\lim_{n\to\infty} \|[p_n, x]\| = \lim_{n\to\infty} \|p_n x - x p_n\| = 0$$

for all $x \in B$.

If A is a separable C*-algebra, a representation π of A on a separable Hilbert space $\mathcal{H}$ is *quasidiagonal* if $\pi(A)$ is a quasidiagonal algebra of operators on $\mathcal{H}$.

A separable C*-algebra A is *quasidiagonal* if A has a faithful quasidiagonal representation.

An alternate characterization of quasidiagonality for algebras of operators is often useful. This characterization can be taken as the definition of quasidiagonality, even on nonseparable Hilbert spaces. The proof is a straightforward exercise.

I.7.3.2. Proposition. Let B be a concrete C*-algebra of operators on a (separable) Hilbert space $\mathcal{H}$. Then B is a quasidiagonal algebra of operators if and only if, for every $x_1, \ldots, x_n \in B$, $\epsilon > 0$, and finite-rank projection $q \in \mathcal{B}(\mathcal{H})$, there is a finite-rank projection $p \in \mathcal{B}(\mathcal{H})$ with $q \leq p$ and

$$\|[p, x_k]\| = \|p x_k - x_k p\| < \epsilon$$

for $1 \leq k \leq n$.

I.7.3.3. Suppose π is a quasidiagonal representation of A on $\mathcal{H}$, and (p_n) is a sequence of finite-rank projections as in the definition. Let $\mathcal{H}_n = p_n\mathcal{H}$ be the range of p_n; then $\mathcal{H}_n$ is a finite-dimensional subspace of $\mathcal{H}$. Set $\pi_n(x) = p_n\pi(x)p_n$ for $x \in A$. Then π_n is a completely positive contraction from A to the matrix algebra $\mathcal{B}(\mathcal{H}_n)$, unital if A and π are unital. If F is a finite subset of A, then for any $\epsilon > 0$ we have

$$\|\pi_n(xy) - \pi_n(x)\pi_n(y)\| = \|p_n\pi(xy)p_n - p_n\pi(x)p_n\pi(y)p_n\|$$

$$= \|p_n\pi(x)\pi(y)p_n - p_n\pi(x)p_n\pi(y)p_n\| = \|p_n\pi(x)[\pi(y)p_n - p_n\pi(y)]p_n\| < \epsilon$$

for all $x, y \in F$ and all sufficiently large n. Thus the π_n are "asymptotically multiplicative" on A.

We have $\|\pi_n(x)\| \to \|\pi(x)\|$ for every $x \in A$. If π is faithful, then $\|\pi_n(x)\| \to \|x\|$. In particular, if $x \neq 0$, $\pi_n(x) \neq 0$ for sufficiently large n. Thus, if A is quasidiagonal, there is a separating family of almost multiplicative completely positive contractions from A to finite-dimensional matrix algebras (finite-dimensional "almost representations" of A). The converse can also be readily shown, so this is another characterization of quasidiagonality ([**?**]).

I.7.3.4. A quasidiagonal C*-algebra need not be nuclear, and conversely. But nuclear quasidiagonal C*-algebras have particularly nice properties. A separable nuclear quasidiagonal C*-algebra was called an *NF algebra* in [BK97], where a number of equivalent characterizations were obtained. It is easily seen that a unital NF algebra must be stably finite and must have a trace. Conversely, there is no stably finite separable unital nuclear C*-algebra which is known not to be NF, and it is conjectured there are none; indeed, the name "NF algebra" was in part chosen in the hope that it would indeed mean "nuclear stably finite" (the name "NSF algebra" seemed just too shameless to the authors of [BK97]). In fact, it is now known (using the classification theorem!) that a separable unital nuclear UCT algebra with a faithful trace, in particular a unital stably finite separable simple nuclear UCT C*-algebra, is an NF algebra. Since there is no obvious connection between the UCT and quasidiagonality (but see ()), the conjecture seems reasonable.

One of the characterizations of NF algebras in [BK97] is essentially a combination of () and I.7.3.3.:

I.7.3.5. Theorem. Let A be a separable unital C*-algebra. Then A is an NF algebra (i.e. nuclear and quasidiagonal) if and only if, for any finite set $F \subseteq A$ and any $\epsilon > 0$, there exist a finite-dimensional C*-algebra D and ucp maps $\alpha : A \to D$ and $\beta : D \to A$ such that

$$\|\beta \circ \alpha(x) - x\| < \epsilon$$

$$\|\alpha(xy) - \alpha(x)\alpha(y)\| < \epsilon$$

for all $x, y \in F$.

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & A \\ & \mathbb{M}_n \arrow[ur, "\beta"'] & \end{tikzcd}$$

The conclusion differs from the one in I.4.4.15. in that the map α is required to be approximately multiplicative, and I.7.3.3. in the existence of the β. In other words, A is NF if and only if not just the complete order structure but also the multiplication in A can be modeled in finite-dimensional C*-algebras. Another way of viewing this is that there is a faithful family of "almost homomorphisms" from A to finite-dimensional C*-algebras, or of finite-dimensional "almost representations" of A; the β's make the approximation more "internal." The result holds also for nonunital C*-algebras if "ucp" is replaced by "cpc".

We can define two more restricted classes of quasidiagonal C*-algebras:

I.7.3.6. Definition. Let A be a separable C*-algebra.

A is *inner quasidiagonal* if A has a separating family of quasidiagonal irreducible representations.

A is *strongly quasidiagonal* if every representation of A on a separable Hilbert space is quasidiagonal.

The actual definition of inner quasidiagonality in [BK01] is different and more technical; the definitions were shown to be equivalent in [BK11].

I.7.3.7. A strongly quasidiagonal C*-algebra is clearly inner quasidiagonal. Since a (countable) direct sum of quasidiagonal representations is quasidiagonal, an inner quasidiagonal C*-algebra is quasidiagonal. The classes all turn out to be distinct.

I.7.3.8. If A (separable) is quasidiagonal, it follows from () that if π is any faithful representation of A on a (separable) Hilbert space $\mathcal{H}$ with the property that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then π is quasidiagonal. Thus, if A is simple and not elementary, and quasidiagonal, it follows that A is strongly quasidiagonal since every representation of A is faithful and does not hit the compacts. Since an elementary C*-algebra is almost trivially strongly quasidiagonal, every (separable) quasidiagonal simple C*-algebra is strongly quasidiagonal, and in particular inner quasidiagonal.

I.7.3.9. A (separable) C*-algebra which is nuclear and inner quasidiagonal is called a *strong NF algebra* in [BK97] (this is technically not the definition, but is equivalent). Every simple nuclear quasidiagonal C*-algebra is a strong NF algebra. Some characterizations of strong NF algebras are given in [BK97]. Two of the most relevant are (we give only the unital versions):

I.7.3.10. Theorem. Let A be a separable unital C*-algebra. Then A is a strong NF algebra if and only if there is an increasing sequence (E_n) of finite-dimensional *-subspaces of A containing 1, each completely order isomorphic to a finite-dimensional C*-algebra A_n via ϕ_n, and coherent unital completely positive idempotent maps $\psi_n : A \to E_n$, such that the corresponding ucp maps $\alpha_n = \phi_n \circ \psi_n : A \to A_n$ are asymptotically multiplicative.

I.7.3.11. By setting $\beta_n = \phi_n^{-1}$, we obtain the condition of I.7.3.5.; but the condition of I.7.3.10. is considerably stronger. Combining this result with some results of Choi and Effros, an inductive limit result is obtained which seems not too far from the definition of an ASH algebra:

I.7.3.12. Definition. Let A be separable unital C*-algebra. A is an *RF algebra* if there is an increasing sequence (E_n) of finite-dimensional *-subspaces of A containing 1, each completely order isomorphic to a finite-dimensional C*-algebra A_n via ϕ_n, and coherent unital completely positive idempotent maps $\psi_n : A \to E_n$, such that the corresponding ucp maps $\alpha_n = \phi_n \circ \psi_n : A \to A_n$ are *-homomorphisms.

I.7.3.13. An RF algebra is in particular nuclear (a strong NF algebra) and *residually finite-dimensional*, i.e. has a separating family of finite-dimensional representations. It can be shown that any separable nuclear residually finite-dimensional C*-algebra is an RF algebra. In particular, it is easily shown that any subhomogeneous C*-algebra is an RF algebra.

I.7.3.14. Theorem. Every strong NF algebra (in particular, every separable simple nuclear quasidiagonal C*-algebra) is an inductive limit of a sequence of RF algebras.

I.7.3.15. One might hope to sharpen this result to say that every strong NF algebra is an ASH algebra. However, this cannot be true, since a quotient of a strong NF algebra is not necessarily strong NF (in fact every separable nuclear C*-algebra is a quotient of a strong NF algebra, even an RF algebra). However, it is quite possible, but unknown, that every *simple* NF algebra is ASH.

I.7.4. Tracial Rank

The first major step forward used quasidiagonality to give an intrinsic characterization of certain classifiable C*-algebras with sufficiently many projections. The idea is that in a simple unital nuclear quasidiagonal C*-algebra with enough projections, approximation by nice subalgebras can be done in a large corner, with all potential pathology of the algebra pushed down into the small complementary corner.

Finite-Dimensional Approximation in Small Corners

The beginning of this procedure was to show that in the presence of enough small projections, good AF-like behavior in small corners followed from quasidiagonality in simple nuclear C*-algebras:

I.7.4.1. Theorem. Let A be a separable simple nuclear quasidiagonal C*-algebra with (SP), not necessarily unital, and let F be a finite subset of A, and $\epsilon > 0$. Then there is a nonzero projection $p \in A$ and a finite-dimensional C*-subalgebra B of pAp containing p, such that, for all $x \in F$,

(i) $\|[p, x]\| < \epsilon$

(ii) $pxp \in_\epsilon B$ (i.e. there is $y \in B$ with $\|pxp - y\| < \epsilon$).

This result is almost implicit in [BK97] or at least [BK01], and follows easily from the results there and a general fact about complete order embeddings (cf. II.2.2.4.), but S. Popa gets credit for explicitly stating and proving it in [Pop97] and pointing out its importance in the classification program (Popa's result was slightly different, with an apparently stronger existence of projections assumption but with nuclearity not assumed).

Making the Corner Large

I.7.4.2. The corner pAp in the conclusion of I.7.4.1. may be very small in general. But this result gives a way to start. H. Lin [] found a way to do a delicate exhaustion argument in the real rank zero case which could make the corner arbitrarily large (but proper), i.e. make the complementary corner arbitrarily small (but nonzero):

I.7.4.3. Theorem. Let A be a simple unital quasidiagonal C*-algebra of real rank zero, and q a nonzero projection in A. Let F be a finite subset of A, and $\epsilon > 0$. Then there is a nonzero projection $p \in A$ and a finite-dimensional C*-subalgebra B of pAp containing p, such that, for all $x \in F$,

(i) $\|[p, x]\| < \epsilon$

(ii) $pxp \in_\epsilon B$ (i.e. there is $y \in B$ with $\|pxp - y\| < \epsilon$).

(iii) $1 - p \leq q$.

No separability is required for this result.

Tracially AF Algebras

This result naturally leads to a definition:

I.7.4.4. Definition. Let A be a separable unital C*-algebra. Then A has *tracial rank zero*, written $TR(A) = 0$, or is *tracially AF*, written TAF, if for every finite subset F of A, $0 \neq a \in A_+$, and $\epsilon > 0$, there is a projection $p \in A$ and finite-dimensional unital C*-subalgebra B of pAp, such that for all $x \in F$,

(i) $\|[p, x]\| < \epsilon$

(ii) $pxp \in_\epsilon B$ (i.e. there is $y \in B$ with $\|pxp - y\| < \epsilon$).

(iii) $1 - p \precsim a$.

The definition makes sense also for nonseparable C*-algebras, and can be extended to nonunital C*-algebras, but we will stick with the separable unital case.

I.7.4.5. It is fairly easy to see that a TAF algebra is quasidiagonal and has real rank zero (cf. [**?**]), and in particular has a tracial state and is therefore stably finite. The converse holds from I.7.4.3. if A is separable and simple, i.e. a separable simple unital C*-algebra is a TAF algebra if and only if it is quasidiagonal and has real rank zero.

I.7.4.6. Since every ASH algebra is quasidiagonal, it follows that a simple unital ASH algebra is TAF if and only if it has real rank zero. Since such an algebra can have nonzero K_1 and/or torsion in K_0, not every TAF algebra is AF.

I.7.4.7. A simple TAF algebra need not be ASH, or even nuclear. A construction described in [**?**, 3.7.7], very similar to the Goodearl algebra construction of I.3.5.4., shows that if B is any separable unital residually finite-dimensional C*-algebra, then B can be embedded in a simple unital TAF algebra A with unique trace, which is an inductive limit of matrix algebras over B (hence A is nuclear if and only if B is). If $B = C^*(\mathbb{F}_2)$ is the full C*-algebra of the free group on two generators, then B is RFD [Cho80], but B, and hence A, cannot be embedded in a nuclear C*-algebra (i.e. is not exact). But there is no known nuclear simple unital TAF algebra which is not ASH, hence classifiable.

General Tracial Rank

CHRONOLOGY

Here is a timeline of some of the most important advances leading to the Classification Theorem. It was not feasible to include all relevant results here (I have tried to include all in the bibliography). The choice of what to include here is subjective and I apologize to anyone who thinks an omitted advance should have been included. I have primarily included work which broke new ground or, at the other extreme, gave definitive results; the advances noted here were not always necessarily the "main" results of the publications listed. When "classifies" is used for C*-algebras, it is via the Elliott invariant unless noted.

Publication dates usually lag the actual date mathematical work is done by at least a year. It has not always been possible to determine when work was actually done, but we have used submission or announcement dates whenever possible.

[Pretty complete to about 2000 but quite incomplete after that so far.]

1928: von Neumann defines abstract Hilbert spaces [vN30a] and begins the study of operator algebras, proving the Bicommutant Theorem [vN30b].

1935: Murray and von Neumann begin their systematic study of factors with the type classification [MVN36].

1940: Gelfand classifies commutative C*-algebras (I.1.2.4.(ii)) [Gel41], [GfRv46].

1943: Murray and von Neumann prove the uniqueness of the hyperfinite II_1 factor $\mathcal{R}$ and show that non-hyperfinite II_1 factors with separable predual exist [MvN43].

Gelfand and Naimark first describe C*-algebras axiomatically and prove that every C*-algebra has a faithful representation (I.1.2.4.(i)) [GN43]; cf. [GfN94].

1946: Segal explicitly describes the GNS construction and uses the term "C*-algebra" for the first time [Seg47].

1953: The axioms for a C*-algebra reach their final form through work of several authors.

1959: Atiyah and Hirzebruch develop topological K-theory [AH59].

Glimm defines and classifies UHF algebras (I.3.1.1.) [Gli60].

1963: Takesaki defines nuclear C*-algebras via tensor products and obtains their basic properties [Tak64].

1964: Dixmier issues the first edition of [Dix96] and the name "C*-algebra" begins to become standard.

1967: Powers exhibits uncountably many AFD Type III factors [Pow67].

Tomita introduces modular theory for von Neumann algebras (put on a sound mathematical footing by Takesaki in [Tak70]).

Dixmier defines and classifies matroid C*-algebras (I.3.3.62.) [Dix67].

1968: Araki and Woods define some invariants for Type III factors and classify ITPFI factors (I.2.3.28.) [AW69].

1969: McDuff exhibits uncountably many (non-AFD) Type II_1 factors [McD69].

1971: Bratteli defines AF algebras and classifies them via Bratteli diagrams [Bra72].

1972: Connes gives the framework and invariants for classifying Type III factors [Con73].

1973: Brown, Douglas, and Fillmore study extensions of C*-algebras and bring K-theory explicitly into the subject [BDF73].

1974: Elliott introduces dimension groups and classifies AF algebras with them [Ell76b].

Bunce and Deddens describe a family of simple A$\mathbb{T}$ algebras which are not AF [BD75].

1975: Connes, building on work of Krieger, characterizes and classifies injective (AFD) factors with separable predual [Con76], [Kri76].

1976: Cuntz defines the O_n and describes their properties [Cun77a].

Choi and Effros characterize nuclearity in terms of completely positive approximations [CE76a], [CE76b], [CE77b]; see also [Kir77].

1977: Cuntz defines the Cuntz semigroup and describes dimension functions on C*-algebras [Cun78].

1978: Blackadar exhibits the first nonunital projectionless simple C*-algebra [Bla80].

Rieffel shows that the irrational rotation algebras contain nontrivial projections [Rie81].

1979: Effros, Handelman, and Shen characterize dimension groups by the Riesz Interpolation Property [EHS80].

Blackadar exhibits the first unital projectionless simple C*-algebra [Bla81].

Cuntz and Krieger define and describe the O_A [CK80].

Pimsner and Voiculescu obtain the Pimsner-Voiculescu exact sequence for crossed products and compute the K-theory of the irrational rotational algebras [PV80a], [PV80b].

Blackadar and Cuntz show that a stable simple C*-algebra without a dimension function contains an infinite projection [BC82].

1980: Kasparov defines and describes KK-theory [Kas80].

Handelman proves that every stably finite unital C*-algebra has a quasitrace [Han81].

Blackadar and Handelman describe the relation between dimension functions and quasitraces () [BH82].

1981: Rieffel introduces stable rank for C*-algebras and proves cancellation theorems for the irrational rotation algebras [Rie83b], [Rie83a].

1983: Haagerup proves that all nuclear C*-algebras are amenable (the converse was already known) [Haa83].

1985: Rosenberg and Schochet define the bootstrap class and prove the Universal Coefficient Theorem and Künneth theorems for it [RS87].

1986: HAAGERUP completes the classification of AFD factors with separable predual [Haa87].

1987: BLACKADAR poses the Fundamental Comparability Question [Bla88].

1989: BLACKADAR constructs a simple unital AF algebra as the crossed product of a non-AF algebra by $\mathbb{Z}_2$ [Bla90b].

1990: ELLIOTT poses the conjecture (or, perhaps more accurately, poses the question) that the Elliott invariant is a complete isomorphism invariant for separable simple nuclear C*-algebras [Ell93a]. [In classic understatement, the abstract of this paper reads: "Certain separable simple C*-algebras which are inductive limits of Type I C*-algebras ... are shown to be classified by K-theoretical and tracial invariants. The question is left open whether this holds for all such simple C*-algebras."]

BROWN and PEDERSEN define real rank for C*-algebras [BP91].

BLACKADAR and RØRDAM show that every state on $K_0(A)$ for simple unital A comes from a quasitrace [BR92].

PUTNAM shows that the irrational rotation algebras have stable rank 1 [Put90].

1991: ELLIOTT and THOMSEN classify simple unital AI algebras (I.3.5.7.) [Ell93a], [Tho94a], [Tho94b].

ELLIOTT classifies simple unital A$\mathbb{T}$ algebras with real rank zero [Ell93b].

BLACKADAR, KUMJIAN, and RØRDAM introduce approximate divisibility and show that the irrational rotation algebras have real rank zero (previous partial results had been obtained by several authors) [BKR92].

HAAGERUP proves that all quasitraces on unital exact C*-algebras are traces [HT99], [Haa14].

1992: RØRDAM begins the classification of purely infinite UCT algebras beyond the Cuntz and Cuntz-Krieger algebras [Rør93], [Rør95], [ER95], [Rør97].

1993: KIRCHBERG announces structure results for purely infinite UCT C*-algebras [Kir95].

ELLIOTT and EVANS show that irrational rotation algebras are A$\mathbb{T}$ algebras [EE93].

1994: Elliott and Gong classify simple unital AH algebras of real rank zero with base spaces of dimension ≤ 3 (these turn out to be everything?) [EG96].

1995: Kirchberg and Phillips independently give a complete classification of purely infinite simple unital UCT C*-algebras [Kir95], [Phi00].

1996: Nielsen and Thomsen classify simple unital A$\mathbb{T}$ algebras [NT96].

Blackadar and Kirchberg show that (separable) quasidiagonal nuclear C*-algebras are generalized inductive limits of finite-dimensional C*-algebras [BK97].

1997: Villadsen constructs simple AH algebras with perforation in K_0 and with stable rank larger than 1 [Vil98b], [Vil99].

Popa introduces the crucial technique of approximation in corners [Pop97] (the technique originated with Connes for factors).

1998: Jiang and Su construct the Jiang-Su algebra $\mathcal{Z}$ and study its properties [JS99].

Lin defines tracially AF C*-algebras and gives a partial classification, the first classification result in the stably finite case without an AH or ASH hypothesis [Lin01d], [Lin01a].

1999: Lin defines general tracial rank for C*-algebras [Lin01c].

2001: Rørdam constructs a simple unital UCT C*-algebra which is finite but not stably finite [Rør03].

2003: Lin shows that a simple UCT C*-algebra with real rank zero and stable rank one has tracial rank zero and is thus classifiable, in particular is AH with base spaces of dimension ≤ 3, completing the classification of such algebras from [EG96] ([Lin04]).

2005: Toms constructs a simple unital AH algebra with the same Elliott invariant as a UHF algebra but which is not $\mathcal{Z}$-stable, using radius of comparison [Tom05].

2010: Winter shows that separable infinite-dimensional simple unital C*-algebras of finite nuclear dimension are $\mathcal{Z}$-stable, implying (among other things) that $\mathcal{Z}$-stability is equivalent to slow dimension growth for simple unital AH algebras and completing the classification of simple AH algebras with slow dimension growth and simple ASH algebras with (PT) and slow dimension growth [Win12].

2015: Elliott, Gong, Lin, and Niu prove that every unital simple separable UCT C*-algebra with finite decomposition rank such that every trace is quasidiagonal has rational generalized tracial rank at most one, and is thus classifiable.

Tikuisis, White, and Winter prove that all traces on a separable unital UCT C*-algebra are quasidiagonal, completing the proof of the classification theorem (unital version) [TWW17].

Chapter II

C*-Algebras

C*-algebras are the nicest kinds of Banach algebras, and have a beautiful theory with profound applications described in the Introduction. This chapter gives a brief introduction to the theory of C*-algebras. We will touch on only some of the basics of this extensive theory here; see [Bla06] (from which this outline is partly excerpted for the convenience of the reader) and the references therein for a much more comprehensive treatment.

II.1. General Theory

II.1.1. Banach Algebras

A Banach algebra is a Banach space which has a multiplication with nice compatibility properties:

II.1.1.1. Definition. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. A *normed algebra* over $\mathbb{F}$ is an algebra () A over $\mathbb{F}$ with a norm satisfying

$$\|xy\| \leq \|x\|\|y\|$$

for all $x, y \in A$ (*submultiplicative norm*). If A is a normed algebra which is a Banach space (i.e. is complete), A is a *Banach algebra*.

We thus have real Banach algebras and complex Banach algebras. For technical reasons, we will only consider complex Banach algebras in this book.

Joint Continuity of Multiplication

In a Banach space (or just normed vector space), addition and scalar multiplication are jointly continuous. In a normed algebra, multiplication is also jointly continuous:

II.1.1.2. Proposition. Let A be a normed algebra. If $x_n, y_n, x, y \in A$ and $x_n \to x$, $y_n \to y$, then $x_n y_n \to xy$.

PROOF: By the usual argument,

$$\|x_ny_n - xy\| = \|x_ny_n - x_ny + x_ny - xy\| \leq \|x_ny_n - x_ny\| + \|x_ny - xy\| \leq \|x_n\|\|y_n - y\| + \|y\|\|x_n - x\| \ .$$

Since (x_n) converges, it is bounded (). 👍

Completion of a Normed Algebra

II.1.1.3. Similarly, if A is a normed algebra, then its completion () has a natural multiplication making it a Banach algebra. For if (x_n) and (y_n) are Cauchy sequences in A, then, for any n and m,

$$\|x_ny_n - x_my_m\| = \|x_ny_n - x_ny_m + x_ny_m - x_my_m\| \leq \|x_ny_n - x_ny_m\| + \|x_ny_m - x_my_m\|$$

$$\leq \|x_n\|\|y_n - y_m\| + \|y_m\|\|x_n - x_m\| \ .$$

Since Cauchy sequences are bounded (), it follows that (x_ny_n) is a Cauchy sequence.

Unital Banach Algebras and Unitization

II.1.1.4. Examples I.1.2.1. and I.1.2.2. show that a Banach algebra need not have a multiplicative identity. A Banach algebra with a multiplicative identity is a *unital Banach algebra*. The multiplicative identity, if it exists, is generally denoted 1. We usually assume $\|1\| = 1$; while this is not necessarily true in general, the norm can be replaced by an equivalent Banach algebra norm with this property. The unit in a C*-algebra, if there is one, always has norm 1.

If a Banach algebra is not unital, a unit can be added in a systematic way:

II.1.1.5. PROPOSITION. Let A be a nonunital Banach algebra over $\mathbb{F}$. Let $A^\dagger = A \oplus \mathbb{F}$. Define addition and scalar multiplication coordinatewise, and multiplication and a norm by

$$(x, \lambda) \cdot (y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$$

$$\|(x, \lambda)\| = \|x\| + |\lambda| \ .$$

Then $A^\dagger$ is a unital Banach algebra containing a copy of A as an ideal () of codimension 1, called the *unitization* of A. Let $\tilde{A}$ be A if A is unital and $A^\dagger$ if A is nonunital.

The rule for multiplication can be remembered by thinking of (x, λ) as $x + \lambda 1$. (The multiplicative identity is $(0, 1)$.)

PROOF: Checking the algebraic rules for an algebra is routine but tedious, and is left to the reader. It is clear that $\|\cdot\|$ is a norm on $A^\dagger$. For submultiplicativity, if $x, y \in A$, $\lambda, \mu \in \mathbb{F}$,

$$\|(x, \lambda)(y, \mu)\| = \|xy + \lambda y + \mu x\| + |\lambda\mu| \leq \|xy\| + \|\lambda y\| + \|\mu x\| + |\lambda\mu|$$

$$\leq \|x\|\|y\| + |\lambda|\|y\| + |\mu|\|x\| + |\lambda\mu| = (\|x\| + |\lambda|)(\|y\| + |\mu|) \ .$$

The map $x \mapsto (x, 0)$ identifies A with a subalgebra (ideal) of $A^\dagger$. To see that $A^\dagger$ is complete, if (a_n, λ_n) is a Cauchy sequence in $A^\dagger$, then (a_n) is a Cauchy sequence in A and (λ_n) a Cauchy sequence in $\mathbb{F}$. These converge to some a and λ, and $(a_n, \lambda_n) \to (a, \lambda)$. 👍

II.1.1.6. Many aspects of Banach algebra theory are technically simpler for unital Banach algebras; II.1.1.5. allows the nonunital case to be largely reduced to the unital case. We do not want to just restrict to unital Banach algebras, however, since many naturally occurring Banach algebras are nonunital. Many, but not all, nonunital Banach algebras have an approximate unit () which can be used as a substitute.

Invertibility, Ideals, and Quotients

One of the remarkable and appealing aspects of the theory of Banach algebras is the intricate interplay between the topology and the algebraic structure. One of the most basic, and most important, examples of this is the use of power series in Banach algebras to construct inverses (and, later, other elements).

II.1.1.7. Recall that an element x in a unital algebra A is *invertible* if there is a $y \in A$ with $xy = yx = 1$. **Caution:** Since a Banach algebra is not necessarily commutative, for x to be invertible it is not generally enough that there is a y with $yx = 1$ (such an x is *left invertible*), or that there is a z with $xz = 1$ (such an x is *right invertible*); cf. (). However, if x is both left and right invertible, the left and right inverses coincide and x is invertible (and conversely): if $yx = xz = 1$, then

$$y = y(xz) = (yx)z = z \ .$$

II.1.1.8. Theorem. Let A be a unital Banach algebra, and $x \in A$. If the power series

$$\sum_{k=0}^{\infty} x^k$$

(where $x^0 = 1$ by convention) converges in A, the sum is a multiplicative inverse for $1 - x$. If $\|x\| < 1$, then the power series converges to a multiplicative inverse for $1 - x$, and

$$\|(1-x)^{-1}\| \leq \frac{1}{1 - \|x\|} \ .$$

Proof: Suppose the series converges, and let y be the sum. If

$$s_n = \sum_{k=0}^{n} x^k$$

is the partial sum to n, so that $s_n \to y$, we have by a simple calculation that

$$s_n(1-x) = (1-x)s_n = 1 - x^{n+1} \ .$$

The left side converges to $y(1-x) = (1-x)y$, and the right side to 1 since $x^{n+1} \to 0$. For the second statement, since $\|x^k\| \leq \|x\|^k$ for every k by submultiplicativity, we have that

$$\sum_{k=1}^{\infty} \|x^k\|$$

converges by comparison to the geometric series with ratio $\|x\|$, and hence $\sum_{k=0}^{\infty} x^k$ converges (). By the triangle inequality,

$$\left\| \sum_{k=0}^{\infty} x^k \right\| \leq \sum_{k=0}^{\infty} \|x^k\| \leq \sum_{k=0}^{\infty} \|x\|^k = \frac{1}{1-\|x\|} \ .$$

II.1.1.9. Corollary. Let A be a unital Banach algebra, and $u \in A$. If $\|u-1\| < 1$, then u is invertible in A, and

$$\|u^{-1}\| \leq \frac{1}{1-\|u-1\|} \ .$$

Proof: Apply the theorem with $x = 1-u$.

II.1.1.10. Corollary. Let A be a unital Banach algebra, and $u, v \in A$. If u is invertible in A and

$$\|u-v\| < \frac{1}{\|u^{-1}\|}$$

then v is also invertible in A, and

$$\|v^{-1}\| \leq \frac{\|u^{-1}\|}{1-\|u^{-1}\|\|u-v\|} \ .$$

Thus the open ball around u of radius $\frac{1}{\|u^{-1}\|}$ consists entirely of invertible elements. In particular, the set of invertible elements in A is an open set.

Continuity of Inversion

II.1.1.11. If A is a unital algebra, write A^{-1} for the set of invertible elements of A. Note that A^{-1} is a group under multiplication, since a product of invertible elements is invertible.

II.1.1.12. Corollary. Let A be a unital Banach algebra. Then the map $x \mapsto x^{-1}$ is a continuous function from A^{-1} to A^{-1}.

Proof: By the sequential criterion, it suffices to show that if (u_n) is a sequence in A^{-1} converging to $u \in A^{-1}$, then $u_n^{-1} \to u^{-1}$. There is an N such that

$$\|u_n - u\| < \frac{1}{2\|u^{-1}\|}$$

for all $n \geq N$. Thus, for $n \geq N$, we have

$$\|u_n^{-1}\| \leq \frac{\|u^{-1}\|}{1-\|u^{-1}\|\|u_n-u\|} < 2\|u^{-1}\| \ .$$

So, for $n \geq N$,

$$\|u_n^{-1} - u^{-1}\| = \|u_n^{-1}(u - u_n)u^{-1}\| \leq \|u_n^{-1}\|\|u_n - u\|\|u^{-1}\| < 2\|u^{-1}\|^2\|u_n - u\|$$

which goes to 0 as $n \to \infty$. 👍

II.1.1.13. Corollary. If A is a unital Banach algebra, then A^{-1} (with the norm topology) is a topological group () under multiplication.

II.1.1.14. If I is an ideal in a Banach algebra A, it follows from continuity of multiplication that the closure $\bar{I}$ is also an ideal. If I is a proper ideal, it is not immediately obvious that $\bar{I}$ is a proper ideal (i.e. I could be dense in A); in fact, this can happen if A is nonunital (). But it cannot happen if A is a unital Banach algebra:

II.1.1.15. Corollary. Let A be a unital Banach algebra, and I proper ideal of A. Then $\bar{I}$ is also a proper ideal of A.

Proof: An ideal in a unital algebra is a proper ideal if and only if it contains no invertible elements. Thus if I is a proper ideal, it is contained in the set S of noninvertible elements of A. But S is closed in A, so $\bar{I} \subseteq S$ and $\bar{I}$ is a proper ideal. 👍

II.1.2. Spectrum

II.1.2.1. Definition. Let A be a unital Banach algebra, $x \in A$. The *spectrum* of x in A is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible in } A\}$$

II.1.2.2. Note that the spectrum of an element of a Banach algebra is a purely algebraic concept, and the norm plays no role in its definition. We may more generally define the spectrum of an element in any unital algebra over $\mathbb{C}$ in an identical way. (Thus, if $\mathcal{V}$ is a finite-dimensional complex vector space and $T \in \mathcal{L}(\mathcal{V})$, then $\sigma(T)$ is the set of eigenvalues of T.) However, the interplay between the topological and algebraic structure in a Banach algebra has important consequences for the *properties* of the spectrum.

II.1.2.3. Proposition. Let A be a unital Banach algebra, $x, y \in A$. Then

(i) $\sigma_A(x)$ is a nonempty compact subset of the plane.

(ii) $\max\{|\lambda| : \lambda \in \sigma_A(x)\} = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf \|x^n\|^{1/n}$. This number is called the *spectral radius* of x, denoted $r(x)$.

(iii) If f is a polynomial with complex coefficients, then

$$\sigma_A(f(x)) = \{f(\lambda) : \lambda \in \sigma_A(x)\} .$$

(iv) $\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}$.

(v) If B is a Banach subalgebra of A and $x \in B$, then $\sigma_A(x) \cup \{0\} \subseteq \sigma_B(x) \cup \{0\}$ and $\partial(\sigma_B(x)) \cup \{0\} \subseteq \partial(\sigma_A(x)) \cup \{0\}$, where ∂ denotes the topological boundary in $\mathbb{C}$. In particular, the spectral radius of x in B is the same as the spectral radius in A (so the notation $r(x)$ is unambiguous).

The proofs of (iii) and (iv) are simple algebraic computations, and the other parts follow from straightforward applications of the theory of II.1.3., using the fact that the function $\lambda \mapsto (x - \lambda 1)^{-1}$ is holomorphic on the complement of the spectrum of x (called the *resolvent set* of x). See e.g. [Bla] for details.

A simple consequence of (i) is:

II.1.2.4. Corollary. [Gelfand-Mazur] The only (complex) Banach division algebra is $\mathbb{C}$.

II.1.3. Holomorphic Functional Calculus

If f is a polynomial with complex coefficients, without constant term (i.e. $f(0) = 0$), and x is an element of an algebra A, then there is an obvious way to apply f to x to obtain an element $f(x) \in A$. If A is a Banach algebra, there is a very important way of extending this procedure to holomorphic functions, called *functional calculus*.

II.1.3.1. If X is a compact subset of $\mathbb{C}$, denote by $H(X)$ the algebra of functions holomorphic in a neighborhood of X and vanishing at 0 if $0 \in X$, with functions identified if they agree on a neighborhood of X. Functional calculus gives a homomorphism from $H(\sigma_A(x))$ to the Banach subalgebra of A generated by x extending the map for polynomials. The image of f is denoted $f(x)$. The element $f(x)$ can be defined using the Cauchy Integral Formula, but in some cases (e.g. if f is entire) it is also given by a power series. If A is unital, functional calculus is also defined for holomorphic functions not vanishing at zero. Functional calculus has the following properties, which (along with the elementary definition for polynomials) determine it uniquely:

II.1.3.2. Proposition. Let A be a Banach algebra and $x \in A$. Then

(i) For any $f \in H(\sigma_A(x))$, $\sigma_A(f(x)) = \{f(\lambda) : \lambda \in \sigma_A(x)\}$.

(ii) If $f \in H(\sigma_A(x))$ and $g \in H(f(\sigma_A(x))) = H(\sigma_A(f(x)))$, so $g \circ f \in H(\sigma_A(x))$, then $(g \circ f)(x) = g(f(x))$.

(iii) If $f_n, f \in H(\sigma_A(x))$ and $f_n \to f$ uniformly on a neighborhood of $\sigma_A(x)$, then $f_n(x) \to f(x)$.

(iv) If B is a Banach algebra and $\phi : A \to B$ a continuous (bounded) homomorphism, then $\phi(f(x)) = f(\phi(x))$ for any $f \in H(\sigma_A(x))$.

Proof: (i): Suppose A is unital. Let f be analytic on $\mathcal{U}$ containing $\sigma_A(x)$. If $\lambda \in \sigma_A(x)$, then $f(z) - f(\lambda) = (z - \lambda)g(z)$ with g analytic on $\mathcal{U}$; then $f(x) - f(\lambda)1 = (x - \lambda 1)g(x)$, and since $x - \lambda 1$ and $g(x)$ commute and $x - \lambda 1$ is not invertible, $f(\lambda) \in \sigma_A(f(x))$. Conversely, if $\mu \notin \{f(\lambda) : \lambda \in \sigma_A(x)\}$, then $h(z) = (f(z) - \mu)^{-1}$ is analytic on $\{z \in \mathcal{U} : f(z) \neq \mu\}$, which contains $\sigma_A(x)$, and $h(x) = (f(x) - \mu 1)^{-1}$.
(ii)-(iv) are straightforward. 👍

II.1.3.3. An especially important case of functional calculus uses the exponential function $f(z) = e^z$. If x is any element of a unital Banach algebra A, then $f(x)$ is defined and denoted e^x. The element e^x is given by the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. For any x, e^x is invertible, with inverse e^{-x}; $e^{x+y} = e^x e^y$ if x and y commute. If A is a Banach *-algebra, then $(e^x)^* = e^{(x^*)}$. Conversely, if $\sigma_A(x)$ is contained in a simply connected open set not containing 0, let $f(z)$ be a branch of the logarithm holomorphic in a neighborhood of $\sigma_A(x)$; then $y = f(x)$ satisfies $e^y = x$. In particular, if $\|1 - x\| < 1$, then there is a $y \in A$ with $\|y\| < \pi/2$ with $e^y = x$ (use the principal branch of log).

II.1.3.4. If A is a unital Banach algebra, write A^{-1} (also often written $\mathrm{GL}_1(A)$) for the (open) set of invertible elements in A, and A_0^{-1} (or $\mathrm{GL}_1(A)_0$) for the connected component of the identity in A^{-1}. Then

$$exp(A) = \{e^y : y \in A\} \subseteq A^{-1}$$

is path-connected, and by the above the subgroup of A^{-1} generated algebraically by $exp(A)$ is a connected open subgroup, hence equal to A_0^{-1}. In particular, A_0^{-1} is an open subgroup of A^{-1}, and every element of A_0^{-1} is a finite product of exponentials.

II.1.4. Banach *-Algebras and C*-Algebras

II.1.4.1. Definition. An *involution* on a Banach algebra A is a conjugate-linear isometric antiautomorphism of order two, usually denoted $x \mapsto x^*$. In other words, $(x+y)^* = x^* + y^*$, $(xy)^* = y^* x^*$, $(\lambda x)^* = \bar{\lambda} x^*$, $(x^*)^* = x$, $\|x^*\| = \|x\|$ for all $x, y \in A$, $\lambda \in \mathbb{C}$. A *Banach *-algebra* is a Banach algebra with an involution.

An *(abstract) C*-algebra* is a Banach *-algebra A satisfying the *C*-axiom*:

$$\|x^* x\| = \|x\|^2 \text{ for all } x \in A$$

The deceptively simple and innocuous C*-axiom turns out to be extremely powerful, forcing rigid structure on a C*-algebra. For example, it follows that the norm is completely determined by the algebraic structure and is thus unique (II.1.5.5.), that *-homomorphisms of C*-algebras are automatically contractive (II.1.5.6.), and that every C*-algebra can be isometrically represented as a concrete C*-algebra of operators (I.1.2.4.(i)). One obvious, but useful, consequence is that in a C*-algebra, $x^* x = 0 \Rightarrow x = 0$. [In fact, in a C*-algebra, $\sum x_j^* x_j = 0$ implies that each $x_j = 0$.]

II.1.4.2. In many older references, abstract C*-algebras were called *B*-algebras*, with the name "C*-algebra" reserved for concrete C*-algebras. The term "C*-algebra," first introduced in [Seg47] (for concrete C*-algebras, but viewed in a somewhat abstract manner), did not become universal until well after the publication of [Dix96] in 1964 (there were occasional references to "B*-algebras" in the literature at least as late as 1980; see even [Lax02]). According to [DB86, p. 6], the "C" in "C*-algebra" originally meant "closed", and not, as commonly believed, "continuous", although the interpretation as standing for "continuous" is nicely in line with the modern point of view of C*-algebra theory as "noncommutative topology."

The issue of terminology is clouded by the fact that several different (but ultimately equivalent) axiom schemes have been used for C*-algebras over the years. For example, it is easily seen that the C*-axiom, along with submultiplicativity of the norm, implies that the involution is isometric, so it is unnecessary to include isometry of the involution as an axiom. The C*-axiom has sometimes been replaced by the apparently weaker axiom that $\|x^* x\| = \|x^*\| \|x\|$ for all x. It turns out that this weaker axiom, along with the Banach algebra axioms, also implies isometry of the involution (a much harder result), so the weakened axiom is equivalent to the C*-axiom. See [DB86] for details about the C*-algebra axioms.

II.1.5. Norm and Spectrum

II.1.5.1. Let A be a C*-algebra. The self-adjoint elements of A form a closed real vector subspace A_{sa} of A. (Note, however, that A_{sa} is not closed under multiplication unless A is commutative.) If x is any element, then $a = (x + x^*)/2$ and $b = (x - x^*)/2i$ are self-adjoint, and $x = a + ib$; thus $A_{sa} + iA_{sa} = A$. The elements a and b are called the *real* and *imaginary* parts of x. It is obvious from their definitions that $\|a\|, \|b\| \le \|x\|$.

II.1.5.2. It follows from the C*-axiom that every nonzero projection, and hence every nonzero partial isometry (in particular, every unitary), has norm 1. Thus, if x is unitary, then $\sigma_A(x) \subseteq \{\lambda : |\lambda| \le 1\}$. Since $x^{-1} = x^*$ is also unitary and hence $\sigma_A(x^{-1}) = \{\lambda^{-1} : \lambda \in \sigma_A(x)\}$ is also contained in the unit disk, $\sigma_A(x)$ is actually contained in the unit circle.

II.1.5.3. If x is self-adjoint, then e^{ix} is unitary, so

$$\sigma_A(e^{ix}) = \{e^{i\lambda} : \lambda \in \sigma_A(x)\}$$

is contained in the unit circle, i.e. $\sigma_A(x) \subseteq \mathbb{R}$.

II.1.5.4. If x is self-adjoint, then from the C*-axiom $\|x^2\| = \|x\|^2$, and by iteration $\|x^{2^n}\| = \|x\|^{2^n}$ for all n. Thus $r(x) = \lim_{n\to\infty} \|x^{2^n}\|^{2^{-n}} = \|x\|$. More generally, if y is normal, then

$$r(y) = \lim_{n\to\infty} \|y^{2^n}\|^{2^{-n}} = \lim_{n\to\infty} \|(y^*)^{2^n} y^{2^n}\|^{2^{-n-1}}$$

$$= \lim_{n\to\infty} \|(y^*y)^{2^n}\|^{2^{-n-1}} = [r(y^*y)]^{1/2} = \|y^*y\|^{1/2} = \|y\| \ .$$

II.1.5.5. Corollary. If A is a C*-algebra and $x \in A$, then $\|x\| = r(x^*x)^{1/2}$. So the norm on a C*-algebra is completely determined by its algebraic structure and is thus unique.

II.1.5.6. Corollary. If A is a Banach *-algebra, B a C*-algebra, and $\phi : A \to B$ a *-homomorphism, then $\|\phi\| \le 1$, i.e. $\|\phi(x)\| \le \|x\|$ for all $x \in A$.

Proof: If $x \in A$, then $\sigma_B(\phi(x)) \cup \{0\} \subseteq \sigma_A(x) \cup \{0\}$. Thus

$$\|\phi(x)\|^2 = \|\phi(x^*x)\| = r(\phi(x^*x)) \le r(x^*x) \le \|x^*x\| \le \|x\|^2 \ .$$

👍

A *-algebra thus has at most one norm making it a (complete) C*-algebra; if it has one, it has no other (even incomplete) norm satisfying the C*-axiom. [A C*-algebra can have other C*-seminorms; and a *-algebra (e.g. a polynomial ring) can have many different (incomplete) norms satisfying the C*-axiom.]

II.1.5.7. Corollary. Let A and B be C*-algebras, $\phi : A \to B$ an injective *-homomorphism. Then ϕ is isometric, i.e. $\|\phi(x)\| = \|x\|$ for all $x \in A$.

Proof: This is an easy consequence of I.1.2.4.(ii) (the precise form) if A and B are commutative. For the general case, if $x \in A$, then $\|x\|^2 = \|x^*x\|$ and $\|\phi(x)\|^2 = \|\phi(x^*x)\|$, so replacing A and B by $C^*(x^*x)$ and $C^*(\phi(x^*x))$ respectively, the result reduces to the commutative case. 👍

II.1.5.8. Corollary. If B is a unital C*-subalgebra of a unital C*-algebra A and $x \in B$, then $\sigma_B(x) = \sigma_A(x)$. If B is a general C*-subalgebra of a general C*-algebra A, and $x \in B$, then $\sigma_B(x) \cup \{0\} = \sigma_A(x) \cup \{0\}$.

Proof: This is true if $x = x^*$ by II.1.2.3.(v). For general x, if $x - \lambda 1$ is invertible in A, then so are $(x - \lambda 1)^*(x - \lambda 1)$ and $(x - \lambda 1)(x - \lambda 1)^*$. Thus they are invertible in B, so $x - \lambda 1$ is left and right invertible in B. 👍

II.1.5.9. Thus the spectrum of an element x of a (unital) C*-algebra A is intrinsic to x and independent of the containing (unital) C*-algebra. We will just write $\sigma(x)$ for $\sigma_A(x)$ from now on.

II.1.6. Continuous Functional Calculus

I.1.2.4.(ii) allows us to extend the holomorphic functional calculus of II.1.3. to continuous functions of normal elements. The following is an easy corollary of I.1.2.4.(ii), where $C_0(\sigma(x))$ denotes the complex-valued continuous functions on $\sigma(x)$ taking the value 0 at 0 if $0 \in \sigma(x)$. Continuous functional calculus, which allows us to apply continuous functions to algebra elements, is a basic and very powerful tool in analyzing the structure of C*-algebras.

II.1.6.1. Corollary. Let A be a C*-algebra, and x a normal element (i.e. $x^*x = xx^*$) of A. Then $C^*(x)$ is isometrically isomorphic to $C_0(\sigma(x))$ under an isomorphism which sends x to the function $f(t) = t$. If A is unital, then $C^*(x, 1) \cong C(\sigma(x))$.

In fact, polynomials in x and x^* without constant term are uniformly dense in $C^*(x)$, and by the Stone-Weierstrass theorem polynomials in λ and $\bar{\lambda}$ without constant term are dense in $C_0(\sigma(x))$. If p is a polynomial in two variables with complex coefficients and no constant term, then

$$\|p(x, x^*)\| = \max\{|p(\lambda, \bar{\lambda})| : \lambda \in \sigma(x)\}.$$

Thus, if f is a complex-valued function which is continuous on $\sigma(x)$, with $f(0) = 0$ if $0 \in \sigma(x)$, then there is a corresponding element $f(x) \in C^*(x)$. This extended functional calculus has the following properties:

II.1.6.2. Proposition. Let A be a C*-algebra and $x \in A$ a normal element. Then

(i) For any $f \in C_0(\sigma(x))$, $\sigma(f(x)) = \{f(\lambda) : \lambda \in \sigma(x)\}$.

(ii) If $f \in C_0(\sigma(x))$ and $g \in C_0(f(\sigma(x))) = C_0(\sigma(f(x)))$, so $g \circ f \in C_0(\sigma(x))$, then $(g \circ f)(x) = g(f(x))$.

(iii) If $f_n, f \in C_0(\sigma(x))$ and $f_n \to f$ uniformly on $\sigma(x)$, then $f_n(x) \to f(x)$.

(iv) If B is a C*-algebra and $\phi : A \to B$ a *-homomorphism, then $\phi(f(x)) = f(\phi(x))$ for any $f \in C_0(\sigma(x))$.

II.1.6.3. Proposition. Let Y be a compact subset of $\mathbb{C}$, and (f_n) a sequence of elements of $C_0(Y)$ converging uniformly on Y to f. Let A be a C*-algebra, (x_n) a sequence of normal elements of A with $x_n \to x \in A$ and $\sigma(x_n) \subseteq Y$ (so $\sigma(x) \subseteq Y$). Then $f_n(x_n) \to f(x)$.

For the proof, let $\epsilon > 0$, and approximate f uniformly on Y within $\epsilon/4$ by a polynomial p; then

$$\|f_n(x_n) - f(x)\| \leq$$

$$\|f_n(x_n) - f(x_n)\| + \|f(x_n) - p(x_n)\| + \|p(x_n) - p(x)\| + \|p(x) - f(x)\|.$$

If n is large enough that $|f_n - f|$ is uniformly less than $\epsilon/4$ on Y, the first term is less than $\epsilon/4$; the second and fourth terms are $< \epsilon/4$ by choice of p, and the third term goes to 0 as $n \to \infty$ by continuity of addition, multiplication, and involution.

This result is often used when (f_n) is a constant sequence.

From functional calculus we obtain:

II.1.6.4. Corollary. Let A be a C*-algebra, and x a normal element of A. Then

x is self-adjoint if and only if $\sigma(x) \subseteq \mathbb{R}$.

x is unitary if and only if $\sigma(x) \subseteq \{\lambda : |\lambda| = 1\}$.

x is a projection if and only if $\sigma(x) \subseteq \{0, 1\}$.

II.1.7. Positive Elements

II.1.7.1. Definition. Let A be a C*-algebra. An element $x \in A$ is *positive* if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$. The set of positive elements in A is denoted A_+. If $x \in A_+$, we write $x \geq 0$.

Note that by II.1.5.8., the property of being positive is independent of the containing C*-algebra, i.e. if B is a C*-subalgebra of A, then $B_+ = B \cap A_+$.

A positive element of $C_0(X)$ is just a function taking only nonnegative real values. The following facts are obvious from I.1.2.4.(ii):

II.1.7.2. Proposition. Let A be a C*-algebra and $x, y \in A$. Then

(i) If $x \geq 0$ and $-x \geq 0$, then $x = 0$.

(ii) If x is normal, then $x^*x \geq 0$. In particular, if $x = x^*$, then $x^2 \geq 0$.

(iii) If $x \geq 0$, then $\|x\| = \max\{\lambda : \lambda \in \sigma(x)\}$.

(iv) If $x = x^*$ and $\|x\| \leq 2$, then $x \geq 0$ if and only if $\|1 - x\| \leq 1$ (in $\tilde{A}$).

(v) If $x, y \geq 0$ and $xy = yx$, then $x + y$ and xy are positive.

(vi) If $x = x^*$, then there is a unique decomposition $x = x_+ - x_-$, where $x_+, x_- \geq 0$ and $x_+x_- = 0$. We have $x_+, x_- \in C^*(x)$. [$x_+ = f(x)$ and $x_- = g(x)$, where $f(t) = \max(t, 0)$ and $g(t) = -\min(t, 0)$.] Thus every element of A is a linear combination of four positive elements.

(vii) Every positive element of a C*-algebra has a unique positive square root. More generally, if $x \geq 0$ and α is a positive real number, there is a positive element $x^\alpha \in C^*(x)_+$; these elements satisfy $x^\alpha x^\beta = x^{\alpha+\beta}$, $x^1 = x$, and $\alpha \mapsto x^\alpha$ is continuous. If x is invertible x^α is also defined for $\alpha \leq 0$.

(viii) $(x, \lambda) \geq 0$ in $A^\dagger$ if and only if $x = x^*$ and $\lambda \geq \|x_-\|$.

For (vii), set $x^\alpha = g_\alpha(x)$, where $g_\alpha(t) = t^\alpha$. Then $g_{\alpha^{-1}}(x^\alpha) = x$. If $b \in A_+$ with $g_{\alpha^{-1}}(b) = x$, then x commutes with b, hence x^α commutes with b; thus $b = x^\alpha$ in $C^*(x^\alpha, b) \subseteq A$.

The two crucial facts about positive elements are:

II.1.7.3. Proposition. Let A be a C*-algebra. Then

(i) A_+ is a closed cone in A; in particular, if $x, y \geq 0$ then $x + y \geq 0$. [A *cone* C in a real or complex vector space is a subset closed under addition and under scalar multiplication by $\mathbb{R}_+$, often (but not always) with the property that $C \cap (-C) = \{0\}$.]

(ii) If $x \in A$, then $x^*x \geq 0$.

To prove (i), since $\mathbb{R}_+A_+ \subseteq A_+$ it suffices to show that $A_+ \cap B$ is a closed convex set, where $B = B_1(A)$ is the closed unit ball in A. But $A_+ \cap B$ is the intersection of the closed convex sets A_{sa}, B, and $\{x : \|1 - x\| \leq 1\}$ (II.1.7.2.(iv)).

The proof of property (ii) uses the fact that if $x = a + ib \in A$ with $a, b \in A_{sa}$, then $x^*x \geq 0$ if and only if $xx^* \geq 0$ by II.1.2.3.(iv), and

$$x^*x + xx^* = 2a^2 + 2b^2 \geq 0$$

by (i) and II.1.7.2.(ii). If $x^*x = c = c_+ - c_-$ as in II.1.7.2.(vi) and $y = xc_-$, then $-y^*y = c_-^3 \geq 0$, so

$$yy^* = (y^*y + yy^*) + (-y^*y) \geq 0$$

and thus $y^*y \geq 0$. So $y^*y = 0$ by II.1.7.2.(i), $c_- = 0$, $x^*x \geq 0$.

Property (ii) is one of the few significant results in mathematics whose proof (due to Kaplansky) first appeared in the *Mathematical Reviews* [Sch52].

II.1.7.4. Corollary. If A is a C*-algebra and $a \in A_+$, then $x^*ax = (a^{1/2}x)^*(a^{1/2}x) \geq 0$ for any $x \in A$. If $a \leq b$, then $x^*ax \leq x^*bx$.

Approximate Polar Decomposition

Polar decomposition cannot be done in general in a C*-algebra, but there is a weak version which is extremely useful.

II.1.7.5. Proposition. *If A is a C*-algebra, $x \in A$, $a \in A_+$, $x^*x \leq a$, and $0 < \alpha < 1/2$, then there is a $u \in \overline{Aa} \subseteq A$ with $u^*u \leq a^{1-2\alpha}$ (hence $\|u\| \leq \|a^{1/2-\alpha}\|$), $uu^* \leq (xx^*)^{1-2\alpha}$, and $x = ua^\alpha$. In particular, $x = u(x^*x)^\alpha$ for some u with $u^*u = (x^*x)^{1-2\alpha}$ (and $\|u\| = \|(x^*x)^{1/2-\alpha}\|$). So every element of A can be written as a product of n elements of A, for any n.*

The proof consists of showing that the sequence $u_n = x[(1/n) + a]^{-1/2}a^{1/2-\alpha}$ is norm-convergent to an element u with $\|u\| \leq \|a^{1/2-\alpha}\|$ (see [Ped79, 1.4.4] for details). Then, if $y_n = 1 - [(1/n) + a]^{-1/2}a^{1/2}$, we have

$$\|x - u_n a^\alpha\|^2 = \|xy_n\|^2 = \|y_n x^* x y_n\| \leq \|y_n a y_n\| = \|a^{1/2} y_n\|^2 \to 0$$

since $f_n(t) = t^{1/2}[1 - (t/[(1/n) + t])^{1/2}] \to 0$ uniformly on $[0, \|a\|]$ by Dini's Theorem.

II.1.7.6. By applying the proposition to x^* and taking adjoints, we obtain the "right-handed version": if $xx^* \leq a$, then for $0 < \alpha < 1/2$ there is a $v \in \overline{aA} \subseteq A$ with $\|v\| \leq \|a^{1/2-\alpha}\|$ and $x = a^\alpha v$.

II.1.7.7. If $0 \leq z \leq a$ in a C*-algebra A, we cannot in general write $z = a^{1/2}ca^{1/2}$ or $z = x^*ax$ for $x, c \in A$, even if A is commutative [consider $f(t) = t$ and $g(t) = t\,\sin^2(t^{-1})$ in $C([0,1])$], although this can always be done if A is a von Neumann algebra. But there is an approximate version:

II.1.7.8. Corollary. *Let $0 \leq z \leq a$ in a C*-algebra A. Then*

(i) *For any $0 < \alpha < 1/2$ there is a $c \in A_+$ with $z = a^\alpha c a^\alpha$ and $\|c\| \leq \|a^{1-2\alpha}\|$.*

(ii) *There is a bounded sequence (x_n) in A with $x_n^* a x_n \to z$, and a bounded sequence (c_n) in A_+ with $a^{1/2}c_n a^{1/2} \to z$.*

Proof: Take $x = z^{1/2}$ in II.1.7.5., and find for each α a u_α with $z^{1/2} = u_\alpha a^\alpha = a^\alpha u_\alpha^*$ and $\|u_\alpha\| \leq \|a^{1/2-\alpha}\| \leq \max(1, \|a^{1/2}\|)$. Then $z = a^\alpha u_\alpha u_\alpha^* a^\alpha = u_\alpha a^{2\alpha} u_\alpha^*$. Set $\alpha_n = 1/2 - 1/n$, $x_n = u_{\alpha_n}^*$ and $c_n = u_{\alpha_n} u_{\alpha_n}^*$.

II.1.8. Properties of Projections

The order structure on projections in a C*-algebra has nice aspects not shared by general positive elements, summarized in the next proposition:

II.1.8.1. Proposition. If p and q are projections in a C*-algebra A, then the following are equivalent:

(i) $p \leq q$.

(ii) $p \leq \lambda q$ for some $\lambda > 0$.

(iii) $pq = qp = p$.

(iv) $q - p$ is a projection.

If these conditions hold, we say p is a *subprojection* of q.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) are essentially trivial, but (ii) $\Rightarrow$ (iii) is not immediately obvious. This implication is a special case of the following proposition (note that $p \leq \lambda q$ for some $\lambda > 0$ implies $\lambda^{-1} p \leq q$):

II.1.8.2. Proposition. Let A be a C*-algebra, $a \in A_+$, q a projection in A. If $a \leq q$, then $aq = qa = a$.

Proof: We have $0 \leq (1-q)a(1-q) \leq (1-q)q(1-q) = 0$ by **??**. Hence, if $x = a^{1/2}(1-q)$, we have $x^*x = 0$, so $x = 0$, $a^{1/2} = a^{1/2}q$, $a = a^{1/2}(a^{1/2}q) = aq$, $a = a^* = q^*a^* = qa$. 👍

But unlike in $\mathcal{B}(\mathcal{H})$ (or in a von Neumann algebra), the projections in a C*-algebra do not form a lattice in general.

II.1.8.3. Recall the definition of equivalence of projections (). There is a slightly stronger notion of unitary equivalence: projections p and q in a C*-algebra A are *unitarily equivalent* (in A) if there is a unitary $u \in A^\dagger$ with $q = u^*pu$ (so $p = uqu^*$). Unitary equivalence is obviously an equivalence relation, and implies equivalence [if $x = u^*p$, then $x^*x = p$ and $xx^* = q$]. Projections p and q are unitarily equivalent if and only if $p \sim q$ and $(1-p) \sim (1-q)$.

II.1.8.4. Proposition. Let A be a C*-algebra, p, q projections in A. If $\|p - q\| < 1$, then $p \sim q$. In fact, there is a unitary $v(p,q) \in A^\dagger$ with $v(p,q)pv(p,q)^* = q$, so p and q are unitarily equivalent.

Proof: If $x = qp + (1-q)(1-p)$, then $1 - x^*x = 1 - xx^* = (p-q)^2$, so $\|1 - x^*x\| = \|1 - xx^*\| = \|p-q\|^2 < 1$ and x is invertible in $A^\dagger$. Let $v(p,q)$ be the unitary in the polar decomposition of x, i.e. $v(p,q) = x(x^*x)^{-1/2}$, $x = v(p,q)(x^*x)^{1/2}$. We have $xp = qx = qp$, $px^* = x^*q = pq$, so $x^*xp = x^*qx = px^*x$, and so $(x^*x)^{1/2}$ commutes with p. Since $xpx^{-1} = qxx^{-1} = q$, we obtain $v(p,q)pv(p,q)^* = q$. 👍

The $v(p,q)$ have additional properties [Bla06, II.3.3.4].

The next observation is elementary, but useful.

II.1.8.5. Proposition. Let A be a C*-algebra, $x, a \in A$, $a = a^*$. Then

(i) If $a \geq 0$ and $x^*ax = 0$, then $a^{1/2}x = x^*a^{1/2} = ax = x^*a = 0$.

(ii) If $\|a\| \leq 1$ and $x^*ax = x^*x$, then $x^*a = x^*$ and $ax = x$.

Proof: (i): Set $y = a^{1/2}x$. Then $y^*y = 0$, so $y = 0$ by the C*-axiom. Thus $ax = a^{1/2}(a^{1/2}x) = 0$. Taking adjoints, $x^*a^{1/2} = x^*a = 0$ also.
(ii): in $A^\dagger$, $1 - a \geq 0$ and $x^*(1-a)x = 0$, so $x^*(1-a) = (1-a)x = 0$. 👍

II.1.8.6. Corollary. If A is a C*-algebra, p a projection in A, and $a = a^* \in A$, $\|a\| \leq 1$, and $pap = p$, then $pa = ap = p$.

II.1.8.7. Proposition. Let A be a C*-algebra, and p and q projections in $M_n(A)$, with q full (). Then $p \precsim k \cdot q$ for some $k \in \mathbb{N}$.

Proof: [Cun77b] Since p is in $Ped(M_n(A))$ (II.1.10.1.), which is the ideal of $M_n(A)$ generated algebraically by q, there are elements $x_1, \dots, x_k \in M_n(A)$ with $p = \sum_{j=1}^k x_j^* q x_j$ [Bla06, II.5.2.10]. We may assume $x_j = qx_jp$ for all j. Let

$$X \in M_k(M_n(A)) \cong M_{kn}(A)$$

be the matrix with $x_1, \dots, x_k$ in the first column and zeroes elsewhere, $P = diag(p, 0, \dots, 0)$, $Q = diag(q, q, \dots, q)$. Then $X^*QX = P$, so $U = QX$ is a partial isometry with $U^*U = P$. UU^* is a projection, and Q is a unit for UU^*, so $UU^* \leq Q$. 👍

II.1.9. Cuntz Equivalence and Hereditary C*-Subalgebras

We recall the definitions of $\precsim$ and $\approx$ from (), as well as some other closely related ones:

II.1.9.8. Definition. Let A be a C*-algebra, $a, b \in A_+$. Then

(i) $b \ll a$ if $ab = b$.

(ii) $b \precsim a$ if there is a sequence (x_n) in A with $x_n^* a x_n \to b$ (equivalently, for any $\epsilon > 0$ there is an $x \in A$ with $\|x^*ax - b\| < \epsilon$).

(iii) $a \approx b$ if $a \precsim b$ and $b \precsim a$.

Of course, $\precsim$ and $\approx$ depend on the choice of A ($\ll$ is independent of the choice of the containing C*-algebra). These relations are transitive; $\precsim$ is reflexive, and $\approx$ is an equivalence relation. [It is not entirely trivial that $\precsim$ is transitive, since the sequence (x_n) cannot be chosen to be bounded in general; but if $b \precsim a$ and $c \precsim b$, and y satisfies $\|y^*by - c\| < \epsilon/2$, choose x with $\|x^*ax - b\| < \epsilon/2\|y\|^2$; then $\|y^*x^*axy - c\| < \epsilon$.] It is easy to see that if p and q are projections, this definition of $p \precsim q$ is equivalent to the previous one.

The relation $\approx$ is called *Cuntz equivalence* and is used in the Cuntz semigroup ().

II.1.9.9. If $\|b\| \leq 1$, then $b \ll a \Longrightarrow b \leq a$. We have $a \ll a$ if and only if a is a projection. If p and q are projections, then $p \ll q$ if and only if $p \leq q$. If $b \ll a$, then a and b obviously commute.

II.1.9.10. By II.1.7.8., $b \leq a \Rightarrow b \precsim a$. But $\precsim$ is a much weaker relation: $x^*ax \precsim a$ for any x, and if $b \in \overline{aAa}$, then $b \precsim a$, so $\overline{aAa} = \overline{bAb} \Rightarrow a \approx b$. In particular, $a \approx \alpha a \approx a^\alpha$ for any $\alpha > 0$. Also, by II.1.7.5. $x^*x \approx xx^*$ for any x. The relations $\precsim$ and $\approx$ measure the "width" (size of the support) of an element rather than its "height" or "location". The relation $b \precsim a$ roughly means that the hereditary C*-subalgebra generated by b is "smaller" than the one generated by a.

II.1.9.11. Proposition. If $a, b, c \in A_+$, $b \ll a$, $\|a\| = 1$, and $\|a - c\| = \eta < 1$, then $(1-\eta)b \leq b^{1/2}cb^{1/2} \leq (1+\eta)b$, so $b^{1/2}cb^{1/2} \approx b$ and $b \precsim c$.

For the proof, note that $b^{1/2}cb^{1/2} = b^{1/2}(c-a)b^{1/2} + b^{1/2}ab^{1/2} \geq -\eta b + b$.

II.1.9.12. Proposition.

(i) If $a, b, b_n \in A_+$, $b_n \precsim a$ for all n, and $b_n \to b$, then $b \precsim a$.

(ii) If $a_1, a_2, b_1, b_2 \in A_+$, $b_1 \precsim a_1$, $b_2 \precsim a_2$, and $a_1 \perp a_2$ (i.e. $a_1a_2 = 0$), then $b_1 + b_2 \precsim a_1 + a_2$.

The proof of (i) is straightforward. For (ii), $b_i \precsim a_i^3$ for $i = 1, 2$, so choose x_n, y_n with $x_n^*a_1^3x_n \to b_1$, $y_n^*a_2^3y_n \to b_2$; then

$$(a_1x_n + a_2y_n)^*(a_1 + a_2)(a_1x_n + a_2y_n) \to b_1 + b_2.$$

II.1.9.13. There are several other very similar relations used in [Cun78], [?], and other references (the same symbols have been used for distinct relations in various references, so the symbols we use do not necessarily coincide with those of the references):

Let A be a C*-algebra, $a, b \in A_+$. Then

$b \preceq a$ if there is a $z \in \tilde{A}$ with $z^*az = b$.

$b \preccurlyeq a$ if there are $x, y \in \tilde{A}$ with $b = xay$.

$b \precapprox a$ if $b \leq z^*az$ for some $z \in \tilde{A}$.

$b \sim a$ if there is an $x \in A$ with $x^*x = a$, $xx^* = b$.

$b \simeq a$ if $b \preceq a$ and $a \preceq b$.

It is obvious that $\preceq$, $\preccurlyeq$, and $\precapprox$ are transitive. The relations $\preceq$, $\preccurlyeq$, and $\precapprox$ are practically the same, although there are nuances of differences between them. We have

$$b \preceq a \Rightarrow b \preccurlyeq a \Rightarrow b \precapprox a \Rightarrow b \precsim a$$

[if $0 \leq b = xay$ for $a \geq 0$, then $b \leq (x^* + y)^*a(x^* + y)$, so $b \preccurlyeq a \Rightarrow b \precapprox a$]. $b \precapprox a \not\Rightarrow b \preccurlyeq a$ (II.1.7.7.); it appears to be unknown whether $b \preccurlyeq a$ implies $b \preceq a$, but is likely false in general. $\precsim$ is a "soft" version of each of these relations.

It can be shown that if A is a von Neumann algebra, then $\preceq$, $\preccurlyeq$, and $\precapprox$ agree on A_+. For general C*-algebras, one can show:

II.1.9.14. Proposition. Let A be a C*-algebra, $a, b \in A_+$. If $b \precapprox a$, then $b^\beta \preceq a^\alpha$ for all α, β, $0 < \alpha < \beta$. In particular, $b \preceq a^\alpha$ for all α, $0 < \alpha < 1$.

Proof: II.1.7.5.

The relation $\sim$ is transitive and thus an equivalence relation, although this is not obvious [**?**]. It is obvious that $\simeq$ is an equivalence relation. We have $a \sim b \Rightarrow a \approx b$ and $a \simeq b \Rightarrow a \approx b$; $\simeq$ is a much "looser" equivalence relation than $\sim$ (e.g. $a \simeq \alpha a$ for any $\alpha > 0$), although $a \sim b$ does not quite imply $a \simeq b$ in general (it does in a von Neumann algebra). The relation $\approx$ is looser yet: we have $a \approx a^2$ for any a, but we rarely have $a \simeq a^2$ (only if a is well-supported, i.e. if $a \simeq p$ for a projection p).

II.1.9.15. There is a version of $\precsim$ for arbitrary elements: we say $x \precsim y$ if there are sequences (z_n), (w_n) with $z_n y w_n \to x$, and $x \approx y$ if $x \precsim y$ and $y \precsim x$. The relation $\preccurlyeq$ also obviously extends to general elements. It is not hard to see that these agree with the previous definitions if $x, y \in A_+$; we have $x \approx x^* \approx x^*x$ for any x. II.1.9.12. remains true in this context, provided that $a_1 \perp a_2$ in the C*-algebra sense, i.e. $a_1a_2 = a_2a_1 = a_1^*a_2 = a_1a_2^* = 0$.

There are some useful functions for comparison theory, as well as other purposes (cf. [Cun78]):

II.1.9.16. Definition. For $\epsilon > 0$, let f_ϵ be the continuous function on $\mathbb{R}$ which is 0 on $(-\infty, \epsilon/2]$, 1 on $[\epsilon, \infty)$, and linear on $[\epsilon/2, \epsilon]$.

There is ambiguity in the literature about the definition of f_ϵ; in some references, what we have called f_ϵ here is called $f_{\epsilon/2}$.

II.1.9.17. If $x \in A_+$, then the elements $f_\epsilon(x)$ have the following properties:

$0 \leq f_\epsilon(x)$ for all ϵ; $\|f_\epsilon(x)\| = 1$ if $\|x\| \geq \epsilon$.

$f_\epsilon(x) \ll f_\delta(x)$ for $0 < \delta \leq \epsilon/2$.

If $\epsilon_n \to \epsilon$ and $x_n \to x$, then $f_{\epsilon_n}(x_n) \to f_\epsilon(x)$ (II.1.6.3.). In particular, $\epsilon \mapsto f_\epsilon(x)$ is continuous.

$f_\epsilon(x) = xg_\epsilon(x) = g_\epsilon(x)^{1/2}xg_\epsilon(x)^{1/2} = x^{1/2}g_\epsilon(x)x^{1/2}$, where $g_\epsilon(t) = t^{-1}f_\epsilon(t)$; in particular, $f_\epsilon(x) \leq \|g_\epsilon(x)\|x$ and $f_\epsilon(x) \precsim x$.

If $h_\epsilon(t) = tf_\epsilon(t)$ for $t \geq 0$, then $h_\epsilon(x) = xf_\epsilon(x)$ and $f_\epsilon(x) = g_{\epsilon/2}(x)h_\epsilon(x)$, so $h_\epsilon(x) \simeq f_\epsilon(x)$ and hence $h_\epsilon(x) \approx f_\epsilon(x)$; as $\epsilon \to 0$,

$$h_\epsilon(x) = xf_\epsilon(x) = x^{1/2}f_\epsilon(x)x^{1/2} = f_\epsilon(x)^{1/2}xf_\epsilon(x)^{1/2} \to x.$$

II.1.9.18. It is also useful to consider the functions ℓ_ϵ, where $\ell_\epsilon(t) = \max(0, t-\epsilon)$; if $x \in A_+$, then $\ell_\epsilon(x) = (x - \epsilon 1)_+ \in A_+$ (computed in $\tilde{A}$). Then $\epsilon \mapsto \ell_\epsilon(x)$ is continuous, $\|\ell_\epsilon(x)\| = \max(0, \|x\| - \epsilon)$, and $\ell_\epsilon(x) \to x$ as $\epsilon \to 0$. We have $\|\ell_\epsilon(x) - h_\epsilon(x)\| \leq \epsilon$, and there are nonnegative continuous functions ϕ and ψ vanishing at 0 (depending on ϵ) such that $\ell_\epsilon = \phi h_\epsilon$ and $h_\epsilon = \psi \ell_\epsilon$, so $\ell_\epsilon(x) = \phi(x)h_\epsilon(x)$, $h_\epsilon(x) = \psi(x)\ell_\epsilon(x)$, and in particular $\ell_\epsilon(x) \simeq h_\epsilon(x) \simeq f_\epsilon(x)$.

The main technical virtue of the ℓ_ϵ is that $\ell_\delta \circ \ell_\epsilon = \ell_{\delta+\epsilon}$ for $\delta, \epsilon > 0$. There is no such simple relationship for the f_ϵ or h_ϵ, although $f_\delta(f_\epsilon(x)) \simeq f_\gamma(x)$ for $\gamma = \epsilon + \epsilon\delta$ (if $\delta \leq 1$).

II.1.9.19. If x is well-supported, then for any sufficiently small ϵ, $f_\epsilon(x^*x)$ and $f_\epsilon(xx^*)$ are the left and right support projections of x respectively. The elements $f_\epsilon(a)$ behave like "generalized support projections" of an element $a \in A_+$ and can often be used as a substitute for an (in general nonexistent, at least in A) actual support projection.

We have the following corollary of II.1.9.11.:

II.1.9.20. Proposition. If $x_n \to x$ in A_+, then, for any $\epsilon > 0$, $f_\epsilon(x_n) \precsim f_{\epsilon/2}(x)$ and $f_\epsilon(x) \precsim f_{\epsilon/2}(x_n)$ for all sufficiently large n.

This result can be improved to show that for any $0 < \delta < \epsilon$, $f_\epsilon(x_n) \precsim f_\delta(x)$ and $f_\epsilon(x) \precsim f_\delta(x_n)$ for all sufficiently large n [KR00, 2.5]. The proof uses the following version of II.1.9.11.:

II.1.9.21. Proposition. [**?**, 2.2] If $x, y \in A_+$, $\|x - y\| < \epsilon$, then $f_\epsilon(x) \precapprox y$.

Proof: If $\delta = \|x - y\|$, then $x - \delta 1 \leq y$ (in $\tilde{A}$). So

$$(\epsilon - \delta) f_\epsilon(x) \leq f_\epsilon(x)^{1/2}(x - \delta 1) f_\epsilon(x)^{1/2} \leq f_\epsilon(x)^{1/2} y f_\epsilon(x)^{1/2}$$

so $f_\epsilon(x) \leq z^* y z$ with $z = (\epsilon - \delta)^{-1/2} f_\epsilon(x)^{1/2}$. 👍

One can then show:

II.1.9.22. Proposition. [**?**, 2.4] Let A be a C*-algebra, $a, b \in A_+$. The following are equivalent:

(i) $b \precsim a$.

(ii) There are $x_n, y_n \in \tilde{A}$ with $x_n a y_n \to b$.

(iii) There are $x_n, y_n \in A$ with $x_n a y_n \to b$.

(iv) $f_\epsilon(b) \preceq a$ for all $\epsilon > 0$.

(v) For every $\epsilon > 0$, there is a $\delta > 0$ such that $f_\epsilon(b) \preceq f_\delta(a)$.

Hereditary Subalgebras

Cuntz equivalence can be alternately described in terms of hereditary C*-subalgebras.

II.1.9.23. Recall that if A is a C*-subalgebra and $x \in A$, then $Her_x(A)$ is defined to be $\overline{x^*Ax}$, which is the smallest hereditary C*-subalgebra of A containing x^*x. Note that $Her_x(A) = Her_{x^*x}(A) = Her_{|x|}(A)$, where $|x| = (x^*x)^{1/2}$. The element x^*x is a *strictly positive element* of $Her_x(A)$ (), i.e. $((x^*x)^{1/n})$ is an approximate unit for $Her_x(A)$ (technically we should scale x to have norm 1 for this; we could instead say that $(f_\epsilon(x^*x))$ is an approximate unit for $Her_x(A)$).

We introduce a notion closely related to approximate unitary equivalence ():

II.1.9.24. Definition. Let A be a separable C*-algebra, and B and D hereditary C*-subalgebras. Then B and D are *approximately conjugate* in A if there is a sequence (x_n) in the unit ball of A such that the maps ϕ_n defined by $\phi_n(b) = x^*bx$ map B into D and converge pointwise to an isomorphism from B onto D, called an *approximate conjugacy*.

This relation makes sense for nonseparable C*-algebras too, and for general C*-subalgebras, but we will restrict to the case of separable C*-algebras and hereditary C*-subalgebras. The restriction that $\|x_n\| = 1$ for all n can almost certainly be deleted. The relation is not obviously either symmetric or transitive, but can be rephrased in the following symmetric way:

II.1.9.25. Proposition. Let A be a separable C*-algebra, and B and D hereditary C*-subalgebras. Then B and D are approximately conjugate in A if and only if there is a sequence (y_n) in A such that $(y_n^*y_n)$ is an approximate unit for B and $(y_ny_n^*)$ is an approximate unit for D.

Proof: First note that if (x_n) is a sequence as in the definition of approximate conjugacy, there is a sequence (z_n) also satisfying the definition with $z_n^*z_n \in B$ for all n: let (b_k) be a dense sequence in B, and (h_m) an approximate unit for B. For each n choose m_n such that

$$\|h_{m_n}b_k - b_k\| < \frac{1}{n}$$

for $1 \leq k \leq n$. Then $z_n = x_nh_{m_n}$ is the required sequence. 👍

The next result gives the prototype example of an approximate conjugacy:

II.1.9.26. Proposition. Let A be a C*-algebra, and $x \in A$. Then x defines an approximate conjugacy from $Her_x(A) = Her_{x^*x}(A)$ onto $Her_{x^*}(A) = Her_{xx^*}(A)$.

For the proof, we need the following fact about polar decomposition in a C*-algebra (cf. [Bla06, III.5.2.16]. If A is a C*-algebra, then A^{**} has a natural structure as a W*-algebra, hence has polar decomposition of all elements. In the following, A^{**} can be replaced by $\pi(A)''$ for any faithful representation of A. Recall that if $x \in A$, then $|x| = (x^*x)^{1/2}$.

II.1.9.27. Proposition. Let A be a C*-algebra, $x \in A$, and $x = u|x|$ its polar decomposition in A^{**}. If $B = Her_{|x|}(A) = \overline{x^*Ax}$ is the hereditary C*-subalgebra of A generated by $|x|$, and $b \in B$, then $ub \in A$, and $(ub)^*(ub) = b^*b$.

Proof: Suppose first that $b = p(|x|)$ for a polynomial p without constant term. Then $p(|x|) = |x|q(|x|)$ for a polynomial q, and $q(|x|) \in A^\dagger \subseteq A^{**}$, so

$$up(|x|) = u|x|q(|x|) = xq(|x|) \in A.$$

Next suppose $f \in C_0(\sigma(|x|))$. Then f is the uniform limit on $\sigma(|x|)$ of a sequence (p_n) of polynomials without constant term, and so $uf(|x|) = \lim_n up_n(|x|) \in A$. In particular, $u|x|^\alpha \in A$ for all $\alpha > 0$. Since $(|x|^\alpha)$ is an approximate unit for B, if $b \in B$ we have $ub = \lim_{\alpha \to 0} u|x|^\alpha b \in A$. Also, u^*u is a unit for x^*x, and hence also for $|x|^\alpha$, in A^{**}; so

$$(ub)^*(ub) = \lim_{\alpha \to 0} b^*u^*u|x|^\alpha b = \lim_{\alpha \to 0} b^*|x|^\alpha b = b^*b.$$

👍

II.1.9.28. Thus, using the notation of II.1.9.27., $y_\epsilon = uf_\epsilon(|x|) \in A$ for any $\epsilon > 0$. Since $xx^* = u(x^*x)u^*$, we have $y_\epsilon y_\epsilon^* = f_\epsilon(|x^*|) \in Her_{x^*}(A)$.

We now give the proof of II.1.9.26..

Proof: Let $x = u|x|$ be the polar decomposition of x in A^{**}. Set $x_n = uf_{2^{-n}}(|x|)$. Then $x_n \in A$, and $Her_{x_n}(A) \subseteq Her_x(A)$. Since $f_{2^{-n-1}}(|x|)$ is a unit for $Her_{x_n}(A)$, the map ϕ_{n+1} defined by $\phi_{n+1}(b) = x_{n+1}bx_{n+1}^*$ is a *-isomorphism of $Her_{x_n}(A)$ onto $Her_{x_n^*}(A)$. Since $\cup_n Her_{x_n}(A)$ is dense in $Her_x(A)$ and $\cup_n Her_{x_n^*}(A)$ is dense in $Her_{x^*}(A)$, the result follows. We can similarly multiply on the left by a suitable approximate unit for D to get $z_n z_n^* \in D$ for all n. 👍

II.1.9.29. Theorem. Let A be a separable C*-algebra, and $a, b \in A_+$. Then $a \precsim b$ if and only if $Her_a(A)$ is approximately conjugate to a hereditary C*-subalgebra of $Her_b(A)$.

II.1.9.30. If A is a separable C*-algebra and $a, b \in A_+$, and $Her_a(A)$ and $Her_b(A)$ are approximately conjugate, then $a \approx b$. What about the converse? It is false in general if a and b are projections and A is not stably finite (). But if A is simple and unital, it could be true in general if a and b are not well supported, or if A is stably finite.

The most elegant way of describing approximate conjugacy is via Hilbert modules (cf. [Bla06, II.7]). The following result was essentially proved in [**?**]:

II.1.9.31. Theorem. Let A be a separable C*-algebra, and B and D hereditary C*-subalgebras. Then B and D are approximately conjugate if and only if the right ideals BA and DA (cf. [Bla06, II.5.3.2(iii)]) of A are isomorphic as Hilbert A-modules, where BA and DA are regarded as submodules of A, regarded as a Hilbert A-module by $\langle x, y\rangle = x^*y$.

II.1.9.32. Extending, the Cuntz semigroup $W(A)$ can be regarded as the semigroup of isomorphism classes of Hilbert A-submodules of the Hilbert A-modules A^n for various n. The completed Cuntz semigroup $Cu(A)$ is similarly the semigroup of isomorphism classes of Hilbert A-submodules of the Hilbert A-module $\mathcal{H}_A$, which by the Stabilization Theorem is just the semigroup of isomorphism classes of countably generated Hilbert A-modules. The ordering is that $B \leq D$ if B is isomorphic to a Hilbert A-submodule of D.

II.1.10. The Pedersen Ideal

There is one especially important and well-behaved ideal in any C*-algebra:

II.1.10.1. Theorem. Let A be a C*-algebra. Then there is a dense ideal $Ped(A)$ in A which is contained in every dense ideal of A. $Ped(A)$ is called the *Pedersen ideal* of A, and has the following properties:

(i) $Ped(A)$ contains $A_+^c = A_+^f$, and is the ideal generated by this set, where

$$A_+^c = \{b \in A_+ : \exists a \in A_+ \text{ with } b \ll a\}$$

$$A_+^f = \{f(a) : a \in A_+, f \in C_0([0,\infty))_+ \text{ vanishes in a neighborhood of } 0\}.$$

In particular, $Ped(A)$ contains all projections (more generally, all well-supported elements) in A.

(ii) $Ped(A)$ is self-adjoint, hereditary, and spanned by its positive elements; $x^*x \in Ped(A) \iff xx^* \in Ped(A) \iff x \in Ped(A)$.

(iii) If $x_1, \dots, x_n \in Ped(A)$, then the hereditary C*-subalgebra of A generated by $x_1, \dots, x_n$ is contained in $Ped(A)$. In particular, $C^*(x_1, \dots, x_n) \subseteq Ped(A)$.

See [Ped79, 5.6] for a discussion and proof. To see that $A_+^f = A_+^c$, if $b \ll a$, and $\|b\| = 1$, set $c = f_{1/2}(a)(b+1)/2$; then $c \geq 0$ (note that a and b commute) and $b = f_{1/2}(c)$, so $A_+^c \subseteq A_+^f$, and the other containment is obvious.

II.1.10.2. If A is unital, then $Ped(A) = A$, and this can even happen (but is unusual) if A is nonunital (e.g. if A is algebraically simple). It is difficult, if not impossible, to effectively describe the Pedersen ideal of a general C*-algebra. However, there are two illuminating examples where it can be easily described: $Ped(\mathcal{K}(\mathcal{H}))$ is the ideal of finite-rank operators on $\mathcal{H}$; and if X is a locally compact Hausdorff space, then $Ped(C_0(X)) = C_c(X)$, the ideal of functions of compact support.

II.1.10.3. If A is a C*-algebra and $a \in A_+$, then there is a sequence $a_n \in A_+^c \subseteq Ped(A)_+$ with $a = \sum a_n$ and $\|a_n\| \leq 2^{-n}$ for $n > 1$: let (h_n) be a sequence of nonnegative continuous functions on $(0,\infty)$, vanishing in a neighborhood of 0, with $\sum h_n(t) = t$ for all t and $\|h_n\| \leq 2^{-n}$ for $n > 1$ (e.g. $h_1(t) = tf_{1/4}(t)$, $h_n(t) = t(f_{2^{-n-1}}(t) - f_{2^{-n}}(t))$ for $n > 1$), and set $a_n = h_n(a)$.

II.1.10.4. If $\pi : A \to B$ is a quotient map (surjective *-homomorphism), then $\pi(Ped(A)) = Ped(B)$. This is obvious from II.1.10.1.(i) since $\pi(A_+^f) = B_+^f$.

II.1.10.5. If B is a C*-subalgebra of A, then $Ped(A) \cap B$ is an ideal of B containing B_+^f, and hence it contains $Ped(B)$; but it may be strictly larger than $Ped(B)$ in general (consider the case where A is unital and B is an ideal in A).

II.1.11. Direct Sums, Products, and Ultraproducts

II.1.11.1. If $\{A_1, \cdots, A_n\}$ is a finite set of C*-algebras, there is a natural notion of direct sum or direct product: $A_1 \oplus \cdots \oplus A_n = A_1 \times \cdots \times A_n$ is the ordinary algebraic direct sum with norm $\|(a_1, \cdots, a_n)\| = \max \|a_i\|$.

If $\{A_i : i \in \Omega\}$ is an infinite set of C*-algebras, there are separate notions of direct sum and product which do not coincide with the algebraic ones.

II.1.11.2. Definition.

$$\prod_{i\in\Omega} A_i = \{(a_i) : \|(a_i)\| = \sup_i \|a_i\| < \infty\}$$

$$\bigoplus_{i\in\Omega} A_i = \{(a_i) : \|a_i\| \to 0 \text{ as } i \to \infty\}$$

in the sense that for every $\epsilon > 0$ there are only finitely many i for which $\|a_i\| \geq \epsilon$. $\bigoplus A_i$ is the closure in $\prod A_i$ of the algebraic direct sum (Ω-tuples with only finitely many nonzero entries).

II.1.11.3. It is easily checked that $\prod A_i$ is a C*-algebra, and $\bigoplus A_i$ is a closed ideal in $\prod A_i$. The quotient C*-algebra is an analog of the Calkin algebra which is useful in various constructions.

Ultraproducts

II.1.11.4. If ω is a free ultrafilter on Ω, then

$$J_\omega = \left\{(a_i) \in \prod A_i : \lim_\omega \|a_i\| = 0\right\}$$

is a closed ideal in $\prod A_i$. The quotient is called the *ultraproduct* of the A_i with respect to ω.

II.1.11.5. These constructions are frequently used with $\Omega = \mathbb{N}$ and all A_i the same A. The product of a sequence of copies of A is usually denoted $l^\infty(A)$ and the direct sum $c_0(A)$. The ultraproduct with respect to a free ultrafilter ω on $\mathbb{N}$ is denoted A_ω.

II.1.12. Inductive Limits

We give a more careful construction of inductive limits, which are outlined in ().

II.1.12.1. An *inductive system* of C*-algebras is defined just as in the purely algebraic setting: a collection $\{(A_i, \phi_{ij}) : i, j \in \Omega, i \leq j\}$, where Ω is a directed set, the A_i are C*-algebras, and ϕ_{ij} is a *-homomorphism from A_i to A_j with $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ for $i \leq j \leq k$. Each ϕ_{ij} is norm-decreasing, so there is a naturally induced C*-seminorm on the algebraic direct limit defined by

$$\|a\| = \lim_{j>i} \|\phi_{ij}(a)\| = \inf_{j>i} \|\phi_{ij}(a)\|$$

for $a \in A_i$; the completion of the algebraic direct limit (with elements of seminorm 0 divided out) is a C*-algebra called the *inductive limit* of the system, written $\varinjlim(A_i, \phi_{ij})$, or just $\varinjlim A_i$ if the ϕ_{ij} are understood. There is a natural *-homomorphism ϕ_i from A_i to the inductive limit.

If all the connecting maps are injective (and hence isometric), the algebraic direct limit may be thought of as the "union" of the A_i, and the inductive limit as the completion of this union. In general, the inductive limit of a system (A_i, ϕ_{ij}) may be naturally regarded as a C*-subalgebra of $(\prod A_i)/(\bigoplus A_i)$.

II.1.12.2. If A_0 is a dense *-subalgebra of a C*-algebra A, there is no good relationship between A_0 and the (closed) ideals of A in general; A may have nonzero ideals whose intersection with A_0 is zero. But if A_0 is a union of C*-subalgebras of A (e.g. if $A = \varinjlim A_i$ and $A_0 = A_\infty$ is the image of the algebraic direct limit), then the relationship is very nice:

II.1.12.3. Proposition. Let A be a C*-algebra, $\{A_i\}$ a collection of C*-subalgebras such that $A_0 = \cup A_i$ is a dense *-subalgebra of A. If J is any closed ideal of A, then $J \cap A_0 = \cup_i (J \cap A_i)$ is dense in J. In particular, if $J \cap A_0 = \{0\}$, then $J = \{0\}$.

Proof: Let K be the closure in A of $J \cap A_0$; then K is a closed ideal of A contained in J, and $K \cap A_i = J \cap A_i$ for all i. Let $\pi : A \to A/K$ be the quotient map, and set $\bar{A} = A/K$, $\bar{A}_i = \pi(A_i)$, $\bar{J} = \pi(J)$. Let $\rho : \bar{A} \to \bar{A}/\bar{J}$ be the quotient map. Then $\rho|_{\bar{A}_i}$ is injective since $\bar{J} \cap \bar{A}_i = \{0\}$ for all i; hence ρ is isometric on each $\bar{A}_i$ and hence on all of $\bar{A}$; so ρ is injective, $\bar{J} = \{0\}$, $J = K$. 👍

II.1.12.4. Corollary. Let $A = \varinjlim (A_i, \phi_{ij})$. If each A_i is simple, then A is simple.

Projections, Unitaries, and Equivalence in Inductive Limits

We now show that projections, unitaries, and equivalence in an inductive limit of C*-algebras can be modeled in the algebraic direct limit. Throughout, we consider an inductive system (A_i, ϕ_{ij}) of C*-algebras, and let A_∞ be the algebraic direct limit and A the completion (inductive limit C*-algebra). To avoid some technicalities irrelevant to the classification program, we will assume all the ϕ_{ij} are injective, and hence isometric. The results are true without this assumption, and even in a more general context, cf. [Bla98, 4.5].

II.1.12.5. Proposition. (i) Let p be a projection in A, and $\epsilon > 0$. Then there is a projection p_0 in A_∞ with $\|p - p_0\| < \epsilon$.

(ii) If q is another projection in A with $p \sim q$ in A, then there is a q_0 in A_∞ with $\|q - q_0\| < \epsilon$ and $p_0 \sim q_0$ in A_∞.

Proof: We may notationally regard A_∞ as $\cup_i A_i$; thus we need to find $p_0, q_0 \in A_i$ for some sufficiently large i. If $x \in A_\infty$ (i.e. $x \in A_i$ for sufficiently large i) with $\|x - p\|$ small enough, then $\|x - x^2\|$ will be small, so the spectrum of x will be contained in the union of a small disk around 0 and a small disk around 1; so p_0 can be made from x by holomorphic functional calculus. For (ii), let $z \in A$ with $z^* z = p$, $zz^* = q$. Approximate z closely by an $x \in A_\infty$ (i.e. in A_i for some large i). Then $x^* x$ is close to p and thus is well-supported, so x has polar decomposition $x = ua$ in A_i with u a partial isometry and a close to p, so u is close to z. Thus $q_0 = uu^*$ is a projection in A_i close to q. Increasing i if necessary so that $p_0 \in A_i$, we have p_0 close to $u^* u$, so

$$p_0 \sim u^* u \sim uu^* = q_0$$

in A_i by II.1.8.4.. (We have avoided doing the hard analysis to give exact limits on the approximations, but they can be done arbitrarily closely due to the continuity of multiplication and polar decomposition, and anyway for II.1.8.4. the approximations do not have to be very close!) 👍

II.1.12.6. Proposition. Suppose (A_i, ϕ_{ij}) is a unital inductive system. Notationally identify each A_i with $\phi_i(A_i) \subseteq A_\infty$.

(i) If u is a unitary in A, then for any $\epsilon > 0$ there is a unitary $u_0 \in A_i \subseteq A_\infty$ for some sufficiently large i with $\|u - u_0\| < \epsilon$.

(ii) If v is another unitary in A which is homotopic to u in $\mathcal{U}(A)$, then for some sufficiently large i there is a unitary $v_0 \in A_i$ with $\|v - v_0\| < \epsilon$ and v_0 homotopic to u_0 in $\mathcal{U}(A_i)$.

Proof: (i): Closely approximate u by an element $x \in A_i$ for some sufficiently large i. Then x is invertible in A_i (if $\|x - u\| < 1$). Let u_0 be the unitary in the polar decomposition of x.

(ii): There is a chain $\{w_1, \ldots, w_n\}$ of unitaries in A with $u = w_1$, $v = w_n$, and $\|w_k - w_{k-1}\|$ small for all k. By (i), for some sufficiently large i, there is a chain $\{z_1, \ldots, z_n\}$ of unitaries in A_i with $\|z_k - w_k\|$ small for all k, and hence $\|z_k - z_{k-1}\|$ small for all k (< 2 will do!). Thus the z_k are all homotopic in A_i. Also, the u_0 of (i) is close to z_1, hence homotopic to z_1. Set $v_0 = z_n$. 👍

The next result is the most important consequence for the classification program:

II.1.12.7. Corollary. Let (A_i, ϕ_{ij}) be a unital inductive system of C*-algebras, and $A = \lim_\to(A_i, \phi_{ij})$. Then

(i) $V(A)$ is the algebraic direct limit of $(V(A_i), \phi_{ij*})$, and hence $K_0(A)$ is the algebraic direct limit of $(K_0(A_i), \phi_{ij*})$.

(ii) $K_1(A)$ is the algebraic direct limit of $(K_1(A_i), \phi_{ij*})$.

This result holds also for nonunital A, and for inductive systems with noninjective connecting maps, but additional argument is needed, and II.1.12.7. is all we will need for the classification program.

II.2. Completely Positive Maps

II.2.1. Complete Positivity

II.2.2. Complete Order Embeddings

II.2.2.1. Lemma. Let B be a unital Banach algebra, and $x \in B$. If x is not invertible, but is a limit of invertibles, then for any $\epsilon > 0$ there is a $y \in B$ with $\|y\| = 1$ and $\|xy\| < \epsilon$.

Proof: Let (x_n) be a sequence of invertible elements of B with $x_n \to x$. If $(\|x_n^{-1}\|)$ is bounded (or has a bounded subsequence), say $\|x_n^{-1}\| \leq M$ for all n, we have

$$\|x_n^{-1}x - 1\| = \|x_n^{-1}(x - x_n)\| \leq M\|x - x_n\| \to 0$$

and thus $x_n^{-1}x$ is invertible for sufficiently large n, i.e. x is left invertible. Similarly, x is right invertible, so x is invertible, a contradiction. Thus $\|x_n^{-1}\| \to \infty$. Set $y_n = \frac{x_n^{-1}}{\|x_n^{-1}\|}$; then $\|y_n\| = 1$ and $x_n y_n \to 0$. We have

$$\|xy_n\| = \|x_n y_n + (x - x_n)y_n\| \leq \|x_n y_n\| + \|(x - x_n)y_n\| \leq \|x_n y_n\| + \|x - x_n\| \to 0 \ .$$

👍

II.2.2.2. Lemma. Let B be a unital C*-algebra with (SP), and let $x \in B$ with $\|x\| = 1$ and $1 \in \sigma(x)$. Then for any $\epsilon > 0$ there is a nonzero projection $p \in B$ with $\|xp - p\| < \epsilon$ and $\|px - p\| < \epsilon$ (and hence $\|px - xp\| < 2\epsilon$).

Proof: Let $\epsilon > 0$. We have that $1 \in \partial(\sigma(x))$, so $x - 1$ is not invertible but is a limit of invertibles; thus by II.2.2.1. there is a $y \in B$ with $\|y\| = 1$ and $\|(x-1)y\| < \frac{\epsilon}{3}$. Thus $\|xb - b\| = \|(x-1)b\| < \frac{\epsilon}{3}$, where $b = yy^*$; b is a positive element of norm 1. By spectral theory, there is a positive element a in B with $\|a\| = 1$, $ab = ba$, and $\|ab - a\| < \frac{\epsilon}{3}$. Let p be a nonzero projection in aBa; then $\|bp - p\| < \frac{\epsilon}{3}$. We have

$$\|xp-p\| = \|xbp-bp+bp-p+x(p-bp)\| \le \|(xb-b)p\|+\|bp-p\|+\|x(p-bp)\| \le \|xb-b\|+2\|p-bp\| < \frac{\epsilon}{3}+\frac{2\epsilon}{3} = \epsilon\,.$$

To show that we can also get $\|px - x\| < \epsilon$, we use the ultraproduct argument. Let (p_n) be a sequence of nonzero projections in B with $\|xp_n - p_n\| \to 0$. Let q and z be the images of the sequence (p_n) and the constant sequence (x) in $\ell^\infty(B)/c_0(B)$ respectively; then $\|z\| = 1$, q is a (nonzero) projection, and $zq = q$. Thus $qz^* = q$, and

$$q = q^4 = q(zq)(qz^*)q = qzqz^*q \le qzqz^*q + qz(1-q)z^*q = qzz^*q \le q$$

and so equality holds and $qz(1-q)z^*q = [qz(1-q)][qz(1-q)]^* = 0$, so $qz(1-q) = 0$, $qz = qzq = q$. Thus $\|p_nx - p_n\| \to 0$, i.e. $\|xp_n - p_n\| < \epsilon$ and $\|p_nx - p_n\| < \epsilon$ for sufficiently large n. 👍

II.2.2.3. Lemma. For every $\epsilon > 0$ there is a $\delta > 0$ with the following property: if A is a C*-algebra, q a projection in A and $a, b \in A_+$ with $\|a\| = \|b\| = 1$, $a \le b$, and $\|qaq - q\| < \delta$, then

$$\|bq - q\| = \|(bq - q)^*\| = \|qb - q\| < \frac{\epsilon}{2}$$

and hence $\|qbq - q\| < \epsilon$ and $\|bq - qb\| < \epsilon$. In particular, $\|aq - q\| = \|qa - q\| < \frac{\epsilon}{2}$ and $\|aq - qa\| < \epsilon$.

Proof: A direct argument giving a formula for δ can be given, but we prefer the following argument.

We first prove the exact case ($\epsilon = 0$): if A, q, a, b are as in the hypotheses and $qaq = q$, we show $bq = q$ also. If $qaq = q$, then

$$0 = q(1-a)q = [(1-a)^{1/2}q]^*[(1-a)^{1/2}q]$$

(calculation in $A^\dagger$; note that $a \le 1$), so $(1-a)^{1/2}q = 0$ by the C*-axiom, and thus $(1-a)q = 0$, $aq = q$. Then $qa = (aq)^* = q^* = q$ also, so q commutes with a. (Thus from the functional representation, we obtain $q \le a$, although we do not need this. Compare with II.1.8.2..) We have $q = qaq \le qbq \le q1q = q$, so $qbq = q$. By the same argument, $qb = bq = q$.

For the approximate result, we use the technique of (). Suppose the result is false. Then there is an $\epsilon > 0$ such that, for every n, there is a C*-algebra A_n and elements q_n, a_n, b_n in A_n satisfying the hypotheses, with $\|q_na_nq_n - q_n\| < \frac{1}{n}$ but $\|b_nq_n - q_n\| \ge \frac{\epsilon}{2}$. Let $\pi : \prod A_n \to [\prod A_n]/[\oplus A_n]$ be the quotient map, and $q = (q_n)$, $a = (a_n)$, $b = (b_n)$ corresponding elements of $\prod A_n$. Then $\pi(q)\pi(a)\pi(q) = \pi(q)$, $\pi(a) \le \pi(b)$, $\|\pi(a)\| = \|\pi(b)\| = 1$, but $\pi(b)\pi(q) \ne \pi(q)$, contradicting the first part of the proof. 👍

The next result, a variation of [**?**,], shows that in the presence of enough projections an injective *-homomorphism can be approximately split off a complete order embedding:

II.2.2.4. Proposition. Let A be a finite-dimensional C*-algebra, and B a C*-algebra with (SP). If $\phi : A \to B$ is a complete order embedding, then for any $\epsilon > 0$ there are

(1) a nonzero projection $p \in B$;

(2) an injective unital *-homomorphism $\pi : A \to pBp$;

(3) a completely positive contraction $\psi : A \to (1-p)B(1-p)$

such that $\|\phi - (\pi \oplus \psi)\| < \epsilon$.

Proof: The proof of this proposition is fairly long to write out, but straightforward and low-tech (at least given standard facts about C*-algebras, completely positive maps, and complete order embeddings). First consider the case $A = \mathbb{C}$. Then $\phi(1)$ is a positive element $b \in B$ with norm 1. Let $\epsilon > 0$, and let f be the following continuous function from $[0,1]$ to $[0,1]$:

$$f(t) = \min(1, (1+\epsilon)t)$$

and g a continuous function from $[0,1]$ to $[0,1]$ with $g(1) = 1$ and $fg = g$. Then $\|f(b) - b\| < \epsilon$. Let p be a nonzero projection in the hereditary subalgebra $g(b)Bg(b)$ of B. Then $pf(b) = f(b)p = p$. Define π and ψ by $\pi(\lambda) = \lambda p$, $\psi(\lambda) = \lambda(f(b) - p) = \lambda((1-p)f(b))$. Note that the construction works also for $\|\phi(1)\| < 1$, but only for $\epsilon > 1 - \|\phi(1)\|$.

The case $A = \mathbb{C}^n$ is essentially identical by iteration. Let $e_1, \dots, e_n$ be the coordinate projections, and $b_k = \phi(e_k)$ for $1 \leq k \leq n$. Then the b_k are positive elements of B of norm 1, and $\sum_{k=1}^n b_k \leq 1$. Let $\epsilon > 0$, and f and g as above but for a smaller ϵ (ϵ/n^2 will do). Let p_1 be a nonzero projection under $g(b_1)$. We have

$$\left(1 - \frac{\epsilon}{n^2}\right) p_1 \leq p_1 b_1 p_1$$

$$p_1 \left(\sum_{k=1}^n b_k\right) p_1 \leq p_1$$

and thus $\|p_1 b_k p_1\| < \frac{\epsilon}{n^2}$ for $k \geq 2$, and

$$\|(1-p_1)b_k(1-p_1) - b_k\| < \frac{\epsilon}{n^2}$$

for $k \geq 2$. Thus, if ϕ_1 is defined by

$$\phi_1(\lambda_1, \dots, \lambda_n) = \lambda_1 p_1 + \sum_{k=1}^n \lambda_k (1-p_1) b_k (1-p_1)$$

we have that ϕ_1 is a completely positive and $\|\phi - \phi_1\| < \frac{\epsilon}{n^2}$. This ϕ_1 will not be a complete order embedding in general (it can be chosen to be a complete order embedding if $p_1 B p_1$ is not one-dimensional, e.g. if B is simple and nonelementary, by making p_1 smaller), but $\|(1-p_1)b_k(1-p_1)\| \approx 1$ for $k \geq 2$. Iterate the construction in $(1-p_1)B(1-p_1)$ to get p_k, $2 \leq k \leq n$, and set $p = p_1 + \cdots + p_n$.

By applying this case to the center of a general A, we reduce the general case to the case $A = \mathbb{M}_n$ for some n, with ϕ unital. In this case, using the Arveson Extension Theorem and results of Choi and Effros (cf. [Bla06, II.6.10.13]), there is a ucp map $\theta : B \to \mathbb{M}_n$ with $\theta \circ \phi = \iota_{\mathbb{M}_n}$ and $\theta|_C$ a homomorphism, where C

is the C*-subalgebra of B generated by $\phi(\mathbb{M}_n)$. Let $\{e_{jk}\}$ be the matrix units of $\mathbb{M}_n$. If $x_{jk} = \phi(e_{jk})$, then $\theta(x_{jk}x_{il}) = e_{jk}e_{il} = \delta_{ki}e_{jl}$ for all i, j, k, l. In particular, $\theta(x_{j1}^*x_{i1}) = \delta_{ij}e_{11}$ for all i, j.

Fix $\epsilon > 0$. Since $\mathbb{M}_n$ is semiprojective, there is a $\delta > 0$ such that, whenever $\{æ_{jk} : 1 \leq j, k \leq n\}$ is a set of "approximate matrix units" in a C*-algebra A satisfying $\|æ_{jk}^* - æ_{kj}\| < \delta$, $\|æ_{jk}æ_{kl} - æ_{jl}\| < \delta$, and $\|æ_{jk}æ_{il}\| < \delta$ for all i, j, k, l, $i \neq k$, then there is a set $\{f_{jk}\}$ of exact matrix units in A with $\|æ_{jk} - f_{jk}\| < \epsilon$ for all j, k. (The δ depends on n as well as ϵ, but n is fixed in this discussion.)

We now proceed as follows. We will not specify the exact degree of approximations, but they can be made as close as desired since there are only finitely many approximations to be made and each can be made arbitrarily close by choosing the previous ones close enough. The proof can be followed most easily by supposing that at the first step q_1 can be chosen so that $q_1y_1^*y_1q_1$ is exactly q_1; then all the subsequent approximations are exact equalities.

Let y_1 be a large product in B of the form

$$y_1 = x_{11} \cdots x_{jj}x_{jk}x_{kk} \cdots x_{11}x_{12} \cdots x_{n-1,n}x_{n,n}x_{n,n-1}x_{n-1,n-1}x_{n-1,n-2} \cdots x_{22}x_{21}x_{11}$$

where each x_{jk} appears, the indices of consecutive factors are compatible, and the last factors are as indicated. Since all x_{jk} have norm 1, $\|y_1\| \leq 1$. But since all x_{jk} are in C,

$$\theta(y_1) = e_{11} \cdots e_{jj}e_{jk}e_{kk} \cdots e_{11}e_{12} \cdots e_{n-1,n}e_{n,n}e_{n,n-1}e_{n-1,n-1}e_{n-1,n-2} \cdots e_{22}e_{21}e_{11} = e_{11}$$

so $\|\theta(y_1)\| = 1$. Since θ is a contraction, $\|y_1\| \geq 1$. Therefore $\|y_1\| = 1$.

By the first part of the proof, there is thus a nonzero projection q_1 in $Her(y_1)$ with $q_1y_1^*y_1q_1 \approx q_1$. Since $y_1^*y_1 \leq x_{11}^*x_{11} = x_{11}^2 \leq x_{11}$, we have

$$q_1x_{11} \approx x_{11}q_1 \approx q_1x_{11}q_1 \approx q_1$$

by Lemma II.2.2.3..

Now let y_2 be y_1 with the last factor of x_{11} omitted, so $y_1 = y_2x_{11}$. Then

$$q_1 \approx q_1y_1^*y_1q_1 = q_1x_{11}y_2^*y_2x_{11}q_1 \approx q_1y_2^*y_2q_1$$

and since $y_2^*y_2 \leq [x_{22}x_{21}]^*[x_{22}x_{21}](= x_{12}x_{22}^2x_{21})$, we have

$$q_1[x_{22}x_{21}]^*[x_{22}x_{21}]q_1 \approx q_1[x_{22}x_{21}]^*[x_{22}x_{21}] \approx [x_{22}x_{21}]^*[x_{22}x_{21}]q_1 \approx q_1$$

by Lemma II.2.2.3.. Thus $x_{22}x_{21}q_1$ is approximately a partial isometry with initial projection q_1, so there is a projection q_2 such that $[x_{22}x_{21}]q_1[x_{22}x_{21}]^* \approx q_2$ and $q_2x_{22}x_{21}q_1 \approx x_{22}x_{21}q_1$. We then have $q_2x_{22}x_{21}q_1x_{21}^*x_{22}q_2 \approx q_2$, and since $x_{22}x_{21}q_1x_{21}^*x_{22} \leq x_{22}^2 \leq x_{22}$, we have

$$q_2x_{22}q_2 \approx q_2x_{22} \approx x_{22}q_2 \approx q_2$$

by Lemma II.2.2.3..We then have

$$q_1 + q_2 \approx x_{11}q_1x_{11} + x_{22}q_2x_{22} \leq x_{11}^2 + x_{22}^2 \leq x_{11} + x_{22} = \phi(e_{11} + e_{22})$$

and since $\|e_{11} + e_{22}\| = 1$, $\|q_1 + q_2\| \approx 1$ and q_1 and q_2 are almost orthogonal. Also,

$$q_1x_{21}^*q_2x_{21}q_1 \approx q_1x_{21}^*x_{22}q_2x_{22}x_{21}q_1 = q_1$$

$$q_2 \approx q_2 x_{22} x_{21} q_1 x_{21}^* x_{22} q_2 \approx q_2 x_{21} q_1 x_{21}^* q_2$$

so $q_2 x_{21} q_1$ is approximately a partial isometry with source projection q_1 and range projection q_2.

Now let y_3 be y_2 with the last two factors $x_{22}x_{21}$ removed, so $y_2 = y_3 x_{22} x_{21}$. Since $x_{22}x_{21}q_1x_{21}^*x_{22} \approx q_2$, we have

$$q_2 y_3^* y_3 q_2 \approx x_{22}x_{21}q_1x_{21}^*x_{22}y_3^*y_3x_{22}x_{21}q_1x_{21}^*x_{22} \approx x_{22}x_{21}q_1y_2^*y_2q_1x_{21}^*x_{22} \approx x_{22}x_{21}q_1x_{21}^*x_{22} \approx q_2 \ .$$

We now repeat the argument for y_2 to obtain a projection q_3 which is approximately the range projection of $x_{33}x_{32}q_2$, almost orthogonal to q_1 and q_2, with $q_3 x_{33} \approx x_{33} q_3 \approx q_3$. Continuing the argument n times, we obtain almost orthogonal projections $q_1, \dots, q_n$, and $q_j x_{jj} \approx x_{jj} q_j \approx q_j$ and $q_j x_{j+1,j}^* x_{j+1,j} q_j \approx q_j$ for all j, i.e. q_j is approximately the source projection for $x_{j+1,j} q_j$.

We continue the same argument except that we do not need to generate additional projections; we obtain for each j, k that $q_k x_{jk}^* q_j x_{jk} q_k \approx q_k$, i.e. q_k is the approximate source projection of the almost partial isometry $q_j x_{jk} q_k$ and q_j the approximate range projection.

For $1 \leq j, k \leq n$, set $æ_{jk} = q_j x_{jk} q_k$. Then $\{æ_{jk}\}$ is a set of approximate matrix units in B. Thus there is a set $\{f_{jk}\}$ of exact matrix units in B with $f_{jk} \approx æ_{jk}$ for all j, k. Set

$$p = \sum_{j=1}^{n} f_{jj} \ .$$

Then p almost commutes with $\phi(\mathbb{M}_n)$, so if we set $\pi(e_{jk}) = f_{jk}$ and $\psi(e_{jk}) = (1-p)x_{jk}(1-p)$ for all j, k, we have the desired approximation. 👍

II.2.2.5. If B has no minimal projections, or equivalently under the (SP) hypothesis if B has no nonzero finite-dimensional corners (e.g. if B is simple and nonelementary), then ψ in II.2.2.4.(3) can be taken to be a complete order embedding.

II.2.3. Order Zero Completely Positive Maps

II.2.4. Splitting Off Order Zero Maps

In [CE77a, 7.1] it is shown that an injective unital homomorphism can be split off from a complete order embedding of one finite-dimensional C*-algebra into another. While this is not exactly true in general even for a complete order embedding of a finite-dimensional C*-algebra into a W*-algebra, there is an approximate version for any target algebra: an order zero complete order embedding can be approximately split off from a complete order embedding of a finite-dimensional C*-algebra into any C*-algebra, in the following sense:

II.2.4.1. Theorem. Let A and B be C*-algebras, with A finite-dimensional, and ϕ a complete order embedding of A into B. Then

The proof of the theorem is an approximate version of the proof of the following lemma. Although the lemma will not be directly used in the proof of the theorem, it is helpful to see a clean version of the argument which may be obscured by the technical details of the approximate version. This lemma is a direct generalization of [CE77a, 7.1] since $B = B^{**}$ if B is finite-dimensional, although the proof is (probably necessarily) different.

II.2.4.2. Lemma. Let A and B be C*-algebras, with A finite-dimensional, and ϕ a complete order embedding of A into B. Then there is a (nonzero) projection p in B^{**}, commuting with $\phi(A)$ (regarding $B \subseteq B^{**}$), such that if $\pi = p\phi$ (i.e. $\pi(x) = p\phi(x) = p\phi(x)p$ for $x \in A$), so that $\phi = \pi \oplus (1-p)\phi$, then π is an injective unital homomorphism from A to $pB^{**}p$.

Proof: We first consider the case $A = \mathbb{M}_n$. By passing to a C*-subalgebra, we may assume (for simplicity) that B is generated as a C*-algebra by $\phi(A)$. Then there is a completely positive projection θ of norm one from B onto $\phi(A)$, such that $\rho := \phi^{-1} \circ \theta$ is a homomorphism from B onto $\mathbb{M}_n$ (which may be regarded as an irreducible representation of B), and $\rho \circ \phi$ is the identity on B ([CE76a, 4.1]; cf. [Bla06, II.6.10.13]).

Let $\{e_{jk}\}$ be the standard matrix units of $\mathbb{M}_n$, and let $x_{jk} = \phi(e_{jk})$. Then $\|x_{jk}\| = 1$ and $x_{kj} = x_{jk}^*$, and by Kadison's inequality (),

$$0 \leq x_{kj}x_{jk} = (x_{jk})^* x_{jk} = \phi(e_{jk})^*\phi(e_{jk}) \leq \phi(e_{jk}^* e_{jk}) = \phi(e_{kk}) = x_{kk}$$

for all j, k. Set

$$z = \sum_{j,k=1}^{n} x_{1j}x_{jk}x_{k1} \ .$$

Then z is not obviously positive, but is self-adjoint since the adjoint of each term is another term in the sum. We have $\|z\| \leq n^2$ since each term has norm ≤ 1; on the other hand, $\rho(z) = n^2 e_{11}$, so $\|z\| = n^2$ and $n^2 \in \sigma(z)$. Let p_1 be the spectral projection of z in B^{**} corresponding to $\{n^2\}$; then $p_1 \neq 0$. Then $p_1 z = z p_1 = p_1 z p_1 = n^2 p_1$. We have

$$n^2 p_1 = \sum_{j,k=1}^{n} p_1 x_{1j}x_{jk}x_{k1}p_1$$

and each term has norm ≤ 1; thus in the C*-algebra $p_1 B^{**} p_1$ we have n^2 elements of norm ≤ 1 adding to n^2 times the identity, so each must be the identity (an immediate consequence of the fact that 1 is an extreme point of the unit ball ([Ped79, 1.1.13]; cf. [Bla06, II.3.2.17])), i.e.

$$p_1 x_{1j}x_{jk}x_{k1}p_1 = p_1$$

for all j, k. Taking $j = k = 1$, we have

$$p_1 = p_1 x_{11}^3 p_1 \leq p_1 x_{11} p_1 \leq p_1$$

so $p_1 x_{11} = x_{11} p_1 = p_1$ since $0 \leq x_{11} \leq 1$ (II.1.8.6.). Also, for each j we have

$$p_1 = p_1 x_{1j}x_{jj}x_{j1}p_1 \leq p_1 x_{1j}x_{j1}p_1 \leq p_1$$

so $p_1 x_{1j} x_{j1} p_1 = p_1$, i.e.

$$p_1 x_{1j}x_{j1} = x_{1j}x_{j1}p_1 = p_1$$

since $0 \leq x_{1j}x_{j1} \leq 1$ (II.1.8.6.). So $x_{j1}p_1$ is a partial isometry with source projection p_1. Let $p_j = x_{j1}p_1x_{1j}$ be the range projection of $x_{j1}p_1$; then

$$p_j x_{j1} = x_{j1}p_1x_{1j}x_{j1} = x_{j1}p_1 = p_j x_{j1} p_1 \ .$$

Now fix j and k. We have

$$p_1 = p_1 x_{1k} x_{kj} x_{j1} p_1 x_{1j} x_{jk} x_{k1} p_1$$

$$p_k = p_k x_{k1} p_1 x_{1k} p_k = p_k x_{k1} p_1 x_{1k} x_{kj} x_{j1} p_1 x_{1j} x_{jk} x_{k1} p_1 x_{1k} p_k$$

and, since $p_k x_{k1} p_1 x_{1k} = x_{k1} p_1 x_{1k} p_k = p_k$, this equals

$$p_k x_{kj} x_{j1} p_1 x_{1j} x_{jk} p_k \leq p_k x_{kj} x_{jk} p_k \leq p_k$$

so we have that $p_k x_{kj} x_{jk} p_k = p_k$, i.e. $u_{jk} = x_{jk} p_k$ is a partial isometry with source projection p_k. Since $x_{jk}^* = x_{kj}$, the range projection of u_{jk} is p_j. In particular,

$$p_k = p_k x_{kk}^2 p_k \leq p_k x_{kk} p_k \leq p_k$$

so $p_k x_{kk}^{1/2} = x_{kk}^{1/2} p_k = p_k$ and $p_k x_{kk} = x_{kk} p_k = p_k$, i.e. $u_{kk} = p_k$ for all k.

Now fix j and k, $j \neq k$. We have

$$p_j + p_k = x_{jj}^{1/2} p_j x_{jj}^{1/2} + x_{kk}^{1/2} p_k x_{kk}^{1/2} \leq x_{jj} + x_{kk} = \phi(e_{jj} + e_{kk})$$

and since $\|\phi(e_{jj} + e_{kk})\| = \|e_{jj} + e_{kk}\| = 1$, $\|p_j + p_k\| = 1$ and $p_j \perp p_k$. Thus if $p = p_1 + \cdots + p_n$, p is a projection and $\{u_{jk}\}$ is a set of matrix units in $pB^{**}p$.

We also have, for $j \neq k$,

$$p_j = p_j x_{jj} p_j \leq p_j x_{jj} p_j + p_j x_{kk} p_j = p_j (x_{jj} + x_{kk}) p_j \leq p_j$$

since $0 \leq x_{jj} + x_{kk} \leq 1$, so $p_j x_{kk} p_j = 0$, i.e.

$$0 = p_j x_{kk}^{1/2} = x_{kk}^{1/2} p_j = p_j x_{kk} = x_{kk} p_j \ .$$

Thus, for $i \neq k$, we have

$$0 \leq p_i x_{kj} x_{jk} p_i \leq p_i x_{kk} p_i = 0$$

for any j, so $p_i x_{kj} = 0$. Taking adjoints, $x_{jk} p_i = 0$ for $i \neq k$ and any j. Thus, for any j and k,

$$p x_{jk} = p_j x_{jk} = u_{jk} = x_{jk} p_k = x_{jk} p$$

so p commutes with each x_{jk} and hence with $\phi(\mathbb{M}_n)$. For any e_{jk}, we have $\phi(e_{jk}) = p\phi(e_{jk}) + (1-p)\phi(e_{jk}) = u_{jk} + (1-p)\phi(e_{jk})$, so for any $x \in \mathbb{M}_n$ we have $\phi(x) = p\phi(x) + (1-p)\phi(x) = \pi(x) + (1-p)\phi(x)$, and π is a unital *-homomorphism of $\mathbb{M}_n$ to $pB^{**}p$.

To get the result for general A, write $A = \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_m}$, and let q_i be the unit of the summand $\mathbb{M}_{n_i}$. For each i, let ϕ_i be the restriction of ϕ to the i'th summand. By the first part of the proof, for each i there is a nonzero projection p_i in B^{**} commuting with $\phi_i(\mathbb{M}_{n_i})$ with $\pi_i = p_i \phi_i$ an injective homomorphism from M_{n_i} to $p_i B^{**} p_i$. For each i we have $\phi(q_i) = p_i + r_i$ for $r_i = (1-p_i)\phi_i(q_i) \geq 0$. Thus, if $p = p_1 + \cdots + p_m$,

$$p \leq p + \sum_{i=1}^{m} r_i = \sum_{i=1}^{m} \phi_i(q_i) = \phi(1_A) \leq 1_{B^{**}}$$

so $\|p\| = 1$, the p_i are mutually orthogonal, and p is a projection which has the desired properties. 👍

II.2.4.3. This argument almost works without passing to B^{**} if B is a W*-algebra (or AW*-algebra), since then z has a spectral projection p_1 in B corresponding to $\{n^2\}$. But the problem is that this spectral projection can be 0. In fact, the result is false in general in this case. For example, let $B = L^\infty([0,1], \mathbb{M}_n)$ and define $\phi : \mathbb{M}_n \to B$ by $[\phi(x)](\lambda) = \lambda x$ for $x \in \mathbb{M}_n$, $\lambda \in [0,1]$. Then ϕ is a complete order embedding, but no homomorphism can be split off. This example can be modified to a unital example if $n > 1$ by setting $[\phi(e_{11})](\lambda) = \lambda e_{11} + (1-\lambda)1$ and $[\phi(e_{jk})](\lambda) = \lambda e_{jk}$ for $(j,k) \neq (1,1)$. Thus the assumption that B is finite-dimensional in [CE77a, 7.1] cannot be relaxed to allowing B to be a W*-algebra.

II.2.4.4. Theorem II.2.4.1. will be proved by replacing the spectral projection $p_1 \in B^{**}$ by some positive elements of B made from z by functional calculus, which will function as "approximate spectral projections."

We have the following generalization of the fact that 1 is an extreme point of the closed unit ball of a unital C*-algebra (the case $D = B$):

II.2.4.5. Proposition. Let B be a C*-algebra and D a C*-subalgebra. Let $b_1, \dots, b_n \in \mathcal{B}$, $\|b_k\| \leq 1$ for all k, $\alpha_1, \dots, \alpha_n \in [0,1]$, $\alpha_1 + \cdots + \alpha_n = 1$, $z = \alpha_1 b_1 + \cdots + \alpha_n b_n$ (so $\|z\| \leq 1$). If $zx = xz = x$ for all $x \in D$, then $b_k x = x b_k = x$ for all k and all $x \in D$.

Proof: Let $x, y \in D_+$, $\|x\| = 1$, $\|y \leq 1$, $xy = yx = y$. Then in $B^\dagger$

$$1 = xzx + (1 - x^2) = \sum_{k=1}^{n} \alpha_k [x b_k x + (1 - x^2)] = \sum_{k=1}^{n} \alpha_k [x c_k x + (1 - x^2)]$$

where $c_k = \frac{1}{2}(b_k + b_k^*)$. Then c_k is self-adjoint, $-1 \leq c_k \leq 1$, so $-x^2 \leq x c_k x \leq x^2$, $-1 \leq 1 - 2x^2 \leq x c_k x + (1 - x^2) \leq 1$, and $\|x c_k x_(1 - x^2)\| \leq 1$. Thus 1 is a convex combination of the $x c_k x + (1 - x^2)$, and from the fact that 1 is an extreme point of the closed unit ball of $B^\dagger$ it follows that $x c_k x + (1 - x^2) = 1$ for all k. In particular, $y = y(x c_k x + (1 - x^2)) = y x c_k x = y c_k x = x c_k y$ and $y^2 = y c_k y$ for all k. We thus have $y(1 - c_k)y = 0$, so $(1 - c_k)^{1/2} y = 0$ (note that $1 - c_k \geq 0$), $(1 - c_k)y = 0$, $c_k y = y$, and thus $y c_k = (c_k y)^* = y^* = y$ also. 👍

Chapter III

Properties of the Elliott Invariant

III.1. Existence and Density of Projections

III.1.1. Existence of Projections

Recall the definitions of the properties (∃P), (LP), (SP), (HP), (FS), and (RR0):

III.1.1.1. Definition. Let A be a unital C*-algebra. Then A satisfies

(∃P) if A contains a nontrivial projection.

(LP) if the linear span of the projections in A is dense in A.

(SP) if every nonzero hereditary C*-subalgebra of A contains a nonzero projection.

(HP) if every hereditary C*-subalgebra of A has an approximate unit of projections.

(FS) if the self-adjoint elements of A with finite spectrum are dense in A_{sa}.

(RR0) if the invertible self-adjoint elements of A are dense in A_{sa}.

III.1.1.2. As noted in (), it is trivial that (HP) ⇒ (SP), (FS) ⇒ (LP), and (if $A \neq \mathbb{C}$) (LP) ⇒ (∃P) and (SP) ⇒ (∃P). The simple argument showing that (FS) ⟺ (RR0) is given in I.7.2.29..

III.1.1.3. All these properties except (RR0) make sense also for nonunital C*-algebras, and it is easy to see that if A is nonunital, then A satisfies each of the properties if and only if $A^\dagger$ does. We extend this to say that A has (RR0) if $A^\dagger$ does. The implications of the previous paragraph then hold also in the nonunital case.

III.1.1.4. To prove the main equivalence, it is convenient to have another definition. A self-adjoint element x of a C*-algebra A is *well-supported* if it is invertible or if 0 is an isolated point of its spectrum. An element $x = x^*$ is well-supported if and only if there is a projection p such that $x = px = xp = pxp$ and x is invertible in pAp. See [Bla06, II.3.2.8] for other equivalent conditions.

III.1.1.5. Theorem. Let A be a C*-algebra. Then the following are equivalent:

(i) A has (HP).

(ii) The well-supported self-adjoint elements of A are dense in A_{sa}.

(iii) A has real rank zero.

(iv) A has (FS).

Proof: (i) $\Rightarrow$ (ii): if $x = x^* \in A$, write $x = x_+ - x_-$. If $\epsilon > 0$, let p be a projection in $Her_{f_\epsilon(x_+)}(A)$ which is an approximate unit for $f_\epsilon(x_+)$ (), specifically so $\|pf_\epsilon(x_+) - f_\epsilon(x_+)\|$ is so small that

$$\|pf_\epsilon(x_+)x_+f_\epsilon(x_+)p - f_\epsilon(x_+)x_+f_\epsilon(x_+)\| < \epsilon \ .$$

Since $\|x_+ - f_\epsilon(x_+)x_+f_\epsilon(x_+)\| < 2\epsilon$, we have

$$\|x_+ - px_+p\|$$

$$\leq \|px_+p - pf_\epsilon(x_+)x_+f_\epsilon(x_+)p\| + \|pf_\epsilon(x_+)x_+f_\epsilon(x_+)p - f_\epsilon(x_+)x_+f_\epsilon(x_+)\| + \|x_+ - f_\epsilon(x_+)x_+f_\epsilon(x_+)\| < 5\epsilon \ .$$

Then $f_{\epsilon/2}(x_+)$ is a unit for p, hence

$$p \leq f_{\epsilon/2}(x_+) \leq \frac{2}{\epsilon}x_+$$

i.e. $x_+ \geq \frac{\epsilon}{2}p$. Thus $px_+p \geq \frac{\epsilon}{2}p^3 = \frac{\epsilon}{2}p$, so px_+p is invertible in pAp, i.e. px_+p is well-supported. Similarly, we have a projection $q \in Her_{f_\epsilon(x_-)}(A)$ such that qx_-q is well supported and $\|x_- - qx_-q\| < 5\epsilon$. Since p and q are orthogonal, $px_+p - qx_-q$ is well-supported, and

$$\|x - (px_+p - qx_-q)\| \leq \|x_+ - px_+p\| + \|x_- - qx_-q\| < 10\epsilon \ .$$

(ii) $\Rightarrow$ (iii): if x is well-supported, then $x + \lambda 1$ is invertible for all sufficiently small nonzero λ.

(iii) $\Rightarrow$ (iv): see I.7.2.29..

(iv) $\Rightarrow$ (i): Suppose A has (FS), and B is a hereditary C*-subalgebra of A. We may assume A is unital. If $x_1, \ldots, x_n \in B$ and $\epsilon > 0$, we seek a projection $p \in B$ with $\|(1-p)x_k\| < \epsilon$ for all k. We have

$$\|(1-p)x_k\|^2 = (1-p)x_kx_k(1-p)\| \leq \|(1-p)x(1-p)\| = \|(1-p)x^{1/2}\|^2$$

where $x = \sum_{k=1}^n x_kx_k^*$. Thus it suffices to show that for every $x \in B_+$ and $\epsilon > 0$ there is a projection $p \in B$ with $\|(1-p)x\| < \epsilon$.

So let $x \in B_+$ and $\epsilon > 0$. We may assume $\epsilon < \frac{1}{2}$ and $\|x\| \leq 1$, i.e. $0 \leq x \leq 1$. Fix δ such that $0 < 6\delta < \epsilon - \epsilon^2$, and fix n large enough that $1 - \delta^{2/n} < \delta$. There is a $y \in A_{sa}$ with finite spectrum with $\|x - y\| < \delta$; replacing y by $f(y)$ for a suitable f we may assume $0 \leq y \leq 1$. We may also choose y so that $\|y^{1/n} - x^{1/n}\| < \delta$ (II.1.6.3.). Let q be the spectral projection of y corresponding to the interval $[\delta, 1]$ (since y has finite spectrum, this is a projection in A). We then have $\|(1-q)y\| < \delta$, and $\|(1 - y^{2/n})q\| < \delta$. Thus we have

$$\|(1-q)x\| < 2\delta \quad \text{and} \quad \|q - x^{1/n}qx^{1/n}\| < 3\delta \ .$$

Set $z = x^{1/n}qx^{1/n}$. Then $z \in B$ and $\|z - z^2\| < 6\delta < \epsilon - \epsilon^2 < \frac{1}{4}$. Thus $\sigma(z)$ is contained in $[0, \epsilon] \cup [1 - \epsilon, 1]$ and has a gap around $\frac{1}{2}$, so if f is a continuous function which is 0 on $[0, \epsilon]$ and 1 on $[1 - \epsilon, 1]$, then $f(z)$ is a projection p in B, and $\|z - p\| \leq \epsilon$. We have

$$\|(1 - p)x\| \leq \epsilon + \|(1 - z)x\| < \epsilon + 3\delta + \|(1 - q)x\| < \epsilon + 5\delta < 2\epsilon \ .$$

👍

The implication (iv) ⇒ (i) is essentially due to Elliott [Ell76a, 3.1]; the proof here is due to Elliott, Pedersen, and the author, and appeared in [Ped80].

III.1.1.6. It is obvious that (HP) passes to hereditary C*-subalgebras. In particular, a corner in a C*-algebra of real rank zero also has real rank zero (this has a somewhat simpler direct proof than via III.1.1.5., cf. [**?**, 3.1.8]). It is not obvious, but true, that real rank zero extends to matrix algebras. The slick proof from [**?**, 3.1.8] is based on the following general fact:

III.1.1.7. Proposition. Let R be a unital ring, and let

$$x = \begin{bmatrix} a & c \\ d & b \end{bmatrix} \in M_2(R) \ .$$

If b is invertible in R, then x is invertible in $M_2(R)$ if and only if $a - cb^{-1}d$ is invertible in R.

Proof: We have

$$x = \begin{bmatrix} 1 & cb^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a - cb^{-1}d & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b^{-1}d & 1 \end{bmatrix} \ .$$

The first and third matrices are invertible in $M_2(A)$ with inverses

$$\begin{bmatrix} 1 & -cb^{-1} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ -b^{-1}d & 1 \end{bmatrix}$$

respectively. Thus x is invertible if and only if the diagonal matrix in the middle is invertible. 👍

III.1.1.8. Proposition. Let A be a unital C*-algebra with real rank zero. Then $M_2(A)$ has real rank zero.

Proof: Let

$$x = \begin{bmatrix} a & c \\ c^* & b \end{bmatrix} \in M_2(A)_{sa} \ .$$

Then $a, b \in A_{sa}$. Let $\epsilon > 0$. Let $\tilde{b}$ be an invertible self-adjoint element of A with $\|b - \tilde{b}\| < \epsilon$. Then $a - c\tilde{b}^{-1}c^* \in A_{sa}$, so there is an invertible self-adjoint element w in A with $\|(a - c\tilde{b}^{-1}c^*) - w\| < \epsilon$. Set $\tilde{a} = w + c\tilde{b}^{-1}c^*$. Then $\tilde{a} - c\tilde{b}^{-1}c^* = w$ is invertible in A, so

$$y = \begin{bmatrix} \tilde{a} & c \\ c^* & \tilde{b} \end{bmatrix}$$

is an invertible self-adjoint element of $M_2(A)$. We have

$$\|a - \tilde{a}\| = \|a - w - c\tilde{b}^{-1}c^*\| < \epsilon$$

so $\|x - y\| = \max(\|a - \tilde{a}\|, \|b - \tilde{b}\|) < \epsilon$. 👍

Putting the results together, we get:

III.1.1.9. Proposition. Let A be a (separable) unital C*-algebra of real rank zero. Then any unital C*-algebra stably isomorphic to A (in particular, every matrix algebra over A) also has real rank zero.

Proof: Iterating III.1.1.8., we get that $M_{2^n}(A)$ has real rank zero for all n. But any unital C*-algebra stably isomorphic to A is a corner in such a matrix algebra for some n. 👍

The "separable" in the statement is only to make stable isomorphism well defined; the result that any corner in a matrix algebra over A has real rank zero does not use separability. The "unital" hypothesis is also superfluous if we extend (RR0) to nonunital C*-algebras as in ().

III.1.1.10. Thus (HP) and (FS) are also preserved under stable isomorphism. Property (SP) is also preserved under stable isomorphism: it obviously passes to hereditary C*-subalgebras, and passes to matrix algebras too. This has a quick proof for simple C*-algebras (apply III.1.1.12. below to $A \otimes \mathbb{K}$):

III.1.1.11. Proposition. Let A be a simple C*-algebra, and a and b nonzero elements of A. Then there is an $x \in A$ with $axb \neq 0$.

Proof: If $axb = 0$ for all $x \in A$, then $(xay)(zbw) = 0$ for all $x, y, z, w \in A$. But the closure of the linear span of all elements of the form xay is a closed ideal of A, hence all of A, and similarly for the closure of the linear span of the zbw, a contradiction. 👍

III.1.1.12. Proposition. Let A be a simple C*-algebra, and B and D nonzero hereditary C*-subalgebras. Then B contains a nonzero hereditary C*-subalgebra isomorphic to a hereditary C*-subalgebra of D.

Proof: Let b and d be nonzero positive elements of B and D respectively. There is an $x \in A$ with $y = bxd \neq 0$. We have that $Her_{yy^*}(A) \subseteq Her_b(A) \subseteq B$ and $Her_{y^*y}(A) \subseteq Her_d(A) \subseteq D$.

We claim $Her_{y^*y}(A) \cong Her_{yy^*}(A)$. There is a C*-algebra containing A in which y has polar decomposition $u(y^*y)^{1/2}$ (faithfully represent A on a Hilbert space, or consider A^{**}). Then $uc \in A$ for every $c \in Her_{y^*y}(A)$ (). For each $\epsilon > 0$ set $z_\epsilon = uf_{\epsilon/2}(y^*y)$. Then $c \mapsto z_\epsilon c z_\epsilon^*$ is an isomorphism of $Her_{f_\epsilon(y^*y)}(A)$ onto $Her_{f_\epsilon(yy^*)}(A)$, and these isomorphisms are compatible, so together give an isomorphism from $Her_{y^*y}(A)$ to $Her_{yy^*}(A)$. 👍

III.1.1.13. Properties ($\exists$P) and (LP) are not preserved under stable isomorphism. Property ($\exists$P) obviously extends to matrix algebras, and (LP) does too (not so obvious: see ()). But neither passes to corners. For example, the Jiang-Su algebra $\mathcal{Z}$ does not have ($\exists$P) and *a fortiori* not (LP), but $M_2(\mathcal{Z})$ satisfies ($\exists$P), and even (LP) by [Ped80] since it has a unique trace.

III.1.1.14. Another potentially interesting existence of projections axiom is

(AP) A is generated as a C*-algebra by its projections.

However, this property is always satisfied in simple C*-algebras with ($\exists$P). In fact, we have the following result from [MM98], improving on [Ped80]:

III.1.1.15. Theorem. Let A be a unital C*-algebra. If A contains a projection p for which both p and $1-p$ are full, then A is generated as an algebra by its projections.

III.1.1.16. Lemma. Let A be a unital C*-algebra, L the linear span of the projections in A, and p a projection in A. Then $pL(1-p) = pA(1-p)$.

Proof: Set $M = pL(1-p)$, and let $x \in M$. If u is a unitary in pAp, let $v = u + (1-p)$; then v is a unitary in A. We have $vLv^* = L$, so $vxv^* = ux \in M$. Since the unitaries in pAp span pAp, we have $aM \subseteq M$ for every $a \in pAp$.

Now let $a \in A$, $\|a\| < 1$, and set $b = pa(1-p)$. Then $\|b\| < 1$, so $p - bb^*$ is an invertible positive element in pAp; call its inverse d. Set

$$q = (p - bb^*) + (p - bb^*)^{1/2}b + b^*(p - bb^*)^{1/2} + b^*b \ .$$

Then q is a projection in A, and $pq(1-p) = (p-bb^*)^{1/2}b$. But $q \in L$, so $pq(1-p) \in M$, i.e. $(p-bb^*)^{1/2}b \in M$. Since $d^{1/2}M \subseteq M$, $b = d^{1/2}(p - bb^*)^{1/2}b \in M$. 👍

Proof: We now give the proof of Theorem III.1.1.15.. Let B be the subalgebra of A generated by the projections of A, and let L be the linear span of the projections in A. Then $L \subseteq B$, so $pL(1-p) \subseteq B$. But $pL(1-p) = pA(1-p)$, so $pA(1-p) \subseteq B$. Similarly, $(1-p)Ap \subseteq B$. Thus $(1-p)ApA(1-p) \subseteq B$. Since p is full in A, by I.1.1.2.(viii) every element of A is a finite linear combination of elements of ApA; thus $(1-p)A(1-p) \subseteq B$. Similarly, since $1-p$ is full in A, $pAp \subseteq B$. So

$$A = pAp + pA(1-p) + (1-p)Ap + (1-p)A(1-p) \subseteq B \ .$$

👍

III.1.1.17. Corollary. Let A be a simple unital C*-algebra containing a nontrivial projection. Then A is generated as an algebra by its projections, and in particular A has (AP).

III.1.1.18. This result improves the result of [Ped80] (attributed there to the authors of [APT73]) that the algebra generated by the projections is dense in A. However, the result of [Ped80] applies also for nonunital simple A, where it is the best possible in general (e.g. the subalgebra of $\mathbb{K}$ generated by the projections is the algebra of finite-rank operators). Note that the principal results of both [Ped80] and [MM98] concern the size of the linear span of the projections.

III.1.1.19. Several other existence of projections properties somewhat relevant to classification are discussed in [Bla88] and [Bla94]. Here are some of them, for a C*-algebra A:

(OP) A contains a sequence of mutually orthogonal nonzero projections (equivalently, A contains a strictly increasing sequence of projections).

(NP) A has no minimal nonzero projections.

(DP) Every nonzero projection in A has two orthogonal equivalent nonzero subprojections.

(PT) The projections of A separate the traces of A, i.e. if τ_1 and τ_2 are distinct traces on A, then there is a projection p in A with $\tau_1(p) \neq \tau_2(p)$.

(CP) The convex hull of the projections of A is dense in the unit ball of A_+.

(FU) (A unital) Every unitary in A can be approximated arbitrarily closely by a unitary with finite spectrum.

(FU_0) (A unital) Every unitary in $\mathcal{U}_0(A)$ can be approximated arbitrarily closely by a unitary with finite spectrum.

(FN) Every normal element of A can be approximated arbitrarily closely by a normal element with finite spectrum.

III.1.1.20. Here are some obvious implications:

$$(\mathrm{DP}) \Rightarrow (\mathrm{NP})$$
$$(\mathrm{CP}) \Rightarrow (\mathrm{LP}) \Rightarrow (\mathrm{PT})$$
$$(\mathrm{FN}) \Rightarrow (\mathrm{FS})$$

and, if A is unital,

$$(\mathrm{FN}) \Rightarrow (\mathrm{FU}) \Rightarrow (\mathrm{FU}_0) \Rightarrow (\mathrm{FS})$$

(for the last implication, use the exponential and logarithm functions). If A is unital, A has (OP) if and only if it has a strictly decreasing sequence of projections ($\mathbb{K}$ is a counterexample in the nonunital case). (NP) means every nonzero corner in A has (OP); $\mathbb{K}$ has (OP) but not (NP). (NP) $\Rightarrow$ (OP) if A contains a nonzero projection ((NP) and (DP) are vacuous if A has no nonzero projections).

A unital C*-algebra A with (FU) must have connected unitary group, since a unitary of finite spectrum is in $\mathcal{U}_0(A)$. (FU_0) (sometimes called (Weak FU)) does not imply connected unitary group.

III.1.1.21. It is not obvious that even an AF algebra satisfies (FN); this turns out to be true but quite nontrivial ([Lin97a]; cf. [Lin97b]). Properties (FU_0) and (FN) have been important in LIN's work on classification ().

The next proposition implies that (SP) $\Rightarrow$ (DP) for a simple nonelementary C*-algebra:

III.1.1.22. Proposition. Let A be a simple C*-algebra, and $B \not\cong \mathbb{C}$ a nonzero hereditary C*-subalgebra of A (the condition $B \not\cong \mathbb{C}$ is automatic if A is not elementary). Then there are nonzero hereditary C*-subalgebras C and D of A with $D \subseteq C \subseteq B$, with $C \cong M_2(D)$.

Proof: Since $B \not\cong \mathbb{C}$, it contains two nonzero orthogonal positive elements a and b (). Since B is simple, there is an $x \in B$ with $y = axb \neq 0$ (). Set $D = Her_{y^*y}(B)$ and $C = Her_{y^*y+yy^*}(B)$. 👍

III.1.1.23. No simple unital C*-algebra is known to satisfy (OP) but not (SP). The examples of Villadsen, Rørdam and/or Toms are potential examples, but Rørdam's example and the Villadsen I examples (hence Toms's example) do have (SP) (), [**?**]. The situation with the Villadsen II examples is unclear. A classifiable simple C*-algebra with (OP) satisfies (SP) and hence (DP). A simple unital C*-algebra in which every nonzero projection is infinite satisfies (DP) (III.2.1.2.). Such an example not satisfying (SP) would be very interesting (it would satisfy (SP) if and only if it is purely infinite).

III.1.1.24. Condition (PT) is vacuous if A has no trace or unique trace. Thus, for example, the Jiang-Su algebra $\mathcal{Z}$ satisfies (PT) but not (LP) or even (∃P). $M_2(\mathcal{Z})$ satisfies (LP) by [Ped80], but not (CP) since every convex combination of nonzero projections has trace at least $\frac{1}{2}$, but positive elements of norm 1 can have arbitrarily small trace [Bla88]. Clearly (FS) implies (CP); they are equivalent for classifiable simple C*-algebras, and possibly in general. It at least may be true in general (for simple unital C*-algebras) that (CP) ⇒ (SP); it seems likely that (CP) ⇒ (NP) in general, since for an infinite-dimensional simple unital C*-algebra (CP) implies there are projections of uniformly arbitrarily small trace.

III.1.1.25. Some of the existence of projections properties stronger than real rank zero discussed in [Bla94] are too strong to be useful for separable simple C*-algebras. We define only two here, for a C*-algebra A:

(FS′) Every $a = a^* \in A$ can be arbitrarily approximated by self-adjoint elements with finite spectrum which commute with a.

(MP) Every masa (maximal commutative C*-subalgebra) in A is generated by projections (i.e. is a commutative AF algebra).

Since every self-adjoint element (even every normal element) is contained in a masa, (MP) ⇒ (FS′); the converse is unclear. Property (FS′) clearly implies (FS). Every W*-algebra satisfies (FS′), but UHF algebras fail to have (FS′) [Phi97] (they fail to have (MP) by [Bla90b]), and I am confident no (infinite-dimensional) classifiable simple unital C*-algebra has (FS′) (at least most do not since (FS′) implies (RR0)), and I even conjecture that no infinite-dimensional separable simple unital C*-algebra has (FS′) (this may depend on some set-theoretic assumptions like the Continuum Hypothesis).

III.1.2. Ordering on K_0

III.1.2.1. Definition. (i) A sequence (p_n) of nonzero projections in a (simple unital) C*-algebra A is a *null sequence* if the sequence $(\rho(p_n))$ converges uniformly to 0 in $\mathrm{Aff}(QT(A))$, i.e. if for every $\epsilon > 0$ there is an N such that $\tau(p_n) < \epsilon$ for all $n \geq N$ and all $\tau \in QT(A)$.

(ii) A sequence (p_n) of nonzero projections in a (simple unital) C*-algebra A is a *thin sequence* if for every nonzero projection q in A there is an N such that $p_n \precsim q$ for all $n \geq N$.

III.1.2.2. If A has (DP), then every thin sequence is a null sequence. These conditions are really only interesting if A is stably finite: if A is not stably finite, then every sequence of nonzero projections in A is a null sequence, and if A is purely infinite, then every sequence of nonzero projections in A is a thin sequence.

III.2. The Infinite Case

III.2.1. K_0 of Properly Infinite C*-Algebras

In stark contrast to the stably finite C*-algebras are the properly infinite ones. In this section, we develop ordered K-theory type results for these algebras; these results are due to Cuntz [Cun81], and the exposition is excerpted from [Bla98].

Recall the definition:

III.2.1.1. Definition. A unital C*-algebra is *properly infinite* if it contains two orthogonal projections equivalent to the identity (i.e. it contains two isometries with mutually orthogonal range projections).

A properly infinite C*-algebra is infinite, but an infinite C*-algebra need not be properly infinite. Any quotient of a properly infinite C*-algebra is properly infinite.

III.2.1.2. Proposition. Let A be a simple unital C*-algebra. Then the following are equivalent:

(i) A is infinite.

(ii) A is properly infinite.

(iii) A contains a sequence of mutually orthogonal equivalent nonzero projections.

(iv) A contains a sequence of isometries with mutually orthogonal range projections (i.e. A contains a unital copy of O_∞, and, in particular, a sequence of mutually orthogonal projections equivalent to 1).

(v) A contains a left [or right] invertible element which is not invertible.

Proof: (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (iii) are trivial.

(v) $\Rightarrow$ (i): If x is left invertible, say $yx = 1$, then $1 = x^*y^*yx \leq \|y\|^2 x^*x$, so x^*x is invertible, and $u = x(x^*x)^{-1/2}$ is an isometry. We have that

$$uu^* = x(x^*x)^{-1}x^* \leq \|(x^*x)^{-1}\| xx^*$$

so if x is not invertible (i.e. xx^* is not invertible), then uu^* is not invertible. The argument if x is right invertible is similar (or just note that if x is right invertible, then x^* is left invertible).

(ii) $\Rightarrow$ (iv): Let x_1, x_2 be isometries in A with orthogonal ranges. Then the isometries $\{x_2^n x_1 : n \geq 0\}$ have mutually orthogonal ranges.

(i) $\Rightarrow$ (iii): If x is a proper isometry in A, and $p = 1 - xx^*$, then $(x^n p {x^*}^n)$ is a sequence of mutually orthogonal equivalent nonzero projections.

(iii) $\Rightarrow$ (ii): Let (p_n) be a sequence of mutually orthogonal nonzero equivalent projections. Let $p = p_1$, and u_n a partial isometry with $u_n^* u_n = p$ and $u_n u_n^* = p_n$ for each n; so $p = u_n^* p_n u_n$. Then there are elements

x_j with $1 = \sum_{j=1}^n x_j^* p x_j = \sum_{j=1}^n x_j^* u_j^* p_j u_j x_j$ for some n. If $q = \sum_{j=1}^n p_j$ and $v = \sum_{j=1}^n u_j x_j$, then $1 = v^* q v$, so qv is an isometry with range projection $qvv^*q \leq q$. Similarly, there is a projection equivalent to 1 under $\sum_{j=kn+1}^{(k+1)n} p_j$ for every k. 👍

III.2.1.3. Note that (ii) $\iff$ (iv) $\implies$ (i) $\iff$ (v) $\implies$ (iii) also in unital nonsimple C*-algebras. But the two arrows cannot be reversed in general: the Toeplitz algebra (I.1.6.35.) and $\mathbb{K}+\mathbb{C}1$ are counterexamples. The implication (iii) $\implies$ (i) can also fail in a simple nonunital C*-algebra: $\mathbb{K}$ is a counterexample.

III.2.1.4. Corollary. Let A be a simple C*-algebra, and p an infinite projection in A. Then there is a sequence (p_n) of mutually orthogonal subprojections of p, each equivalent to p. In particular, p is properly infinite. If A is unital, then p contains a subprojection equivalent to 1_A.

Proof: The corner pAp is a simple unital C*-algebra with identity p. For the last statement, if (p_n) is a sequence of mutually orthogonal, mutually equivalent nonzero subprojections of p, then the argument of III.2.1.2.[(iii) $\Rightarrow$ (ii)] shows that 1_A is equivalent to a subprojection of $\sum_{j=1}^n p_j \leq p$ for some n. (In fact, p has an entire sequence of mutually orthogonal subprojections each equivalent to 1_A.) 👍

III.2.1.5. Call a projection in a unital C*-algebra *very full* if it contains a subprojection equivalent to the identity. Any projection equivalent to a very full projection is very full. If A is simple, then a nontrivial projection in A is very full if and only if it is infinite (III.2.1.4.).

III.2.1.6. Proposition. Let A be a properly infinite unital C*-algebra, and $V_f(A)$ the subset of $V(A)$ consisting of the equivalence classes of very full projections of A. Then:

(i) $V_f(A)$ is a subsemigroup of $V(A)$.

(ii) $V_f(A)$ is a group (with a different identity element than $V(A)$).

(iii) The homomorphism from $V(A)$ to $K_0(A)$ induces an isomorphism from $V_f(A)$ onto $K_0(A)$.

Proof: (i): Since A contains a sequence of mutually orthogonal projections equivalent to the identity, it follows easily that every projection in $M_\infty(A)$ is equivalent to a projection in A. So if p and q are very full projections in A, then $p \oplus q$ is equivalent to a projection in A, which is very full since it contains a copy of p. Thus $V_f(A)$ is a subsemigroup.

(ii): Notice first that if p and q are projections in A with p very full, then $q \sim q' \leq p$ such that $p - q'$ is very full, since p contains two mutually orthogonal subprojections equivalent to 1, so q' can be taken to be a copy of q under one of them. In particular, if p is very full, then $p \sim p' \leq p$ with $p - p'$ very full.

If p, q are very full and $p \sim p' \leq p$ with $p - p'$ very full, we claim that $[q] = [p - p'] + [q]$. Let $q \sim q' \leq q$ with $q - q'$ very full. We may assume $p \leq q'$ by replacing it by an equivalent projection; then

$$[q] = [q'] = [q' - p] + [p] = [q' - p] + [p'] = [q' - p + p'].$$

So $[q] + [p - p'] = [q' - p + p'] + [p - p'] = [q'] = [q]$.

Thus $[p - p']$ is an identity for $V_f(A)$ for any very full p, and by the uniqueness of an identity in a semigroup $p - p' \sim q - q'$ for any other very full q.

To get inverses, let p be very full, and let p', p'' be orthogonal subprojections of p, equivalent to p, with $p - p' - p''$ very full. Then $[p] + [p - p' - p''] = [p - p' - p''] + [p''] = [p - p']$, so $[p - p' - p'']$ is the inverse of $[p]$.

(iii): If q is any projection in A and p a very full projection with $[p]$ the identity of $V_f(A)$, then p and q have orthogonal representatives $p', q' \in A$; $p' + q'$ is very full and has the same class as q in $K_0(A)$. Thus the map from $V_f(A)$ to $K_0(A)$ is surjective, since the image is a subgroup containing $K_0(A)_+$. If p is very full and the class of p is 0 in $K_0(A)$, then there is a $p' \in A$ with $p' \sim p$, and a $q \in A$, $q \perp p'$, such that $q \sim p' + q$. But then q is very full, and by cancellation in $V_f(A)$ we get that $[p]$ is the identity of $V_f(A)$, i.e. the map is injective.

III.2.1.7. Corollary. If A is a properly infinite (unital) C*-algebra, in particular if A is an infinite simple unital C*-algebra, then $K_0(A)_+$ is all of $K_0(A)$. In fact, every element of $K_0(A)$ is represented by a projection in A.

III.2.1.8. Corollary. Two infinite projections in a simple unital C*-algebra are equivalent if and only if they have the same K_0-class. Two infinite projections with infinite complements which have the same K_0-class are unitarily equivalent.

III.2.2. Purely Infinite C*-algebras

There is an even stronger notion of infiniteness. Recall the definition of a purely infinite C*-algebra in I.5.1.1..

III.2.2.1. A purely infinite C*-algebra is simple. A corner in a purely infinite C*-algebra is purely infinite. If A is purely infinite, then any matrix algebra over A is isomorphic to a corner in A, and hence is also purely infinite.

Purely infinite simple unital C*-algebras have several equivalent characterizations (I.5.1.3.). Examples of purely infinite simple unital C*-algebras are (countably decomposable) type III factors, the Calkin algebra, the Cuntz algebras (VI.2.2.6.(ii)), and the simple Cuntz-Krieger algebras (VI.2.2.6.(iii)).

III.2.2.2. A unital C*-algebra $A \neq \mathbb{C}$ is purely infinite if and only if, for every nonzero $a, b \in A$, there are $x, y \in A$ with $xay = b$. This can be taken as the definition of a purely infinite C*-algebra in the nonunital case. It is a nontrivial fact [Bla06, V.2.3.12] that a nonunital purely infinite C*-algebra A contains a nonzero projection p; then necessarily pAp is purely infinite (and unital). It then turns out that every nonunital σ-unital purely infinite C*-algebra (by this definition) is the stabilization of a unital purely infinite C*-algebra (a corner), cf. III.2.2.7..

III.2.2.3. An infinite simple unital C*-algebra need not be purely infinite: if A is Rørdam's example (**??**), then $M_2(A)$ is infinite but not purely infinite since it contains a finite projection.

It is conceivable that there is a simple unital C*-algebra in which every nonzero projection is infinite, but in which there is a nonzero projectionless hereditary C*-subalgebra. Such a C*-algebra would not be purely infinite.

III.2.2.4. Proposition. Let A be a purely infinite (unital) C*-algebra, and let $a \in A$, $\|a\| = 1$. Then for any $\epsilon > 0$ there are $x, y \in A$ with $xay = 1$ and $\|x\|, \|y\| \leq 1 + \epsilon$. If $a \geq 0$, x can be chosen to be y^*.

Proof: First suppose $a \geq 0$. Let $\epsilon > 0$, and set $\eta = (1+\epsilon)^2$ and $b = \eta a$. Let f be a continuous function from $[0,1]$ to $[0,1]$ with $f(1) = 1$ and $f = 0$ on $\left[0, \frac{1}{\eta}\right]$. Let p be a projection equivalent to 1 in $f(a)Af(a)$; then $p \leq b$. If u is an isometry in A with $uu^* = p$, i.e. $u^*pu = 1$, we have $z := u^*bu \geq u^*pu = 1$, so z is invertible and $\|z^{-1/2}\| \leq 1$. Set $y = \eta^{1/2}uz^{-1/2} = (1+\epsilon)uz^{-1/2}$; then $\|y\| \leq 1+\epsilon$ and

$$y^*ay = z^{-1/2}u^*buz^{-1/2} = 1 \ .$$

For general a, we have y with $\|y\| \leq 1+\epsilon$ and $y^*(a^*a)y = (y^*a^*)ay = 1$, and $\|y^*a^*\| \leq 1+\epsilon$.

III.2.2.5. Corollary. A (unital) inductive limit of purely infinite C*-algebras is purely infinite.

Proof: Let $A = \lim_{\to}(A_i)$ with each A_i purely infinite (and unital connecting maps). Since each A_i is simple, the connecting maps are injective and we may regard the A_i as unital C*-subalgebras of A, with dense union. Let $a \in A$, $\|a\| = 1$. Choose $b \in A_i$ for some i with $\|b\| = 1$ and $\|a - b\| < \frac{1}{4}$, and let $x, y \in A_i$ with $xby = 1$ and $\|x\|, \|y\| \leq 2$. Then $\|xay - 1\| = \|x(a-b)y\| \leq 4\|a-b\| < 1$, so xay is invertible in A and $[(xay)^{-1}x]ay = 1$.

A purely infinite C*-algebra has (SP) by definition, but more is true:

III.2.2.6. Proposition. [**?**] A purely infinite C*-algebra has real rank zero.

Proof: Let A be a purely infinite simple C*-algebra. Let $x = x^* \in A$, and $0 < \epsilon \ll 1$. Set $y = xf_\epsilon(x^2)$; then y approximates x and there is an $h \in A_+$, $h \neq 0$, $h \perp y$. Let s be a projection in $\overline{hAh}$ which is equivalent to 1_A, and set $p = 1_A - s$. Then $y = pyp$ and $p \precsim s$; let v be a partial isometry in A with $v^*v = p$, $vv^* = q \leq s$, and set $r = s - q$. Then $p + q + r = 1_A$, and

$$z = y + \epsilon(v + v^* + r)$$

is invertible, with

$$z^{-1} = -\epsilon^{-2}vyv^* + \epsilon^{-1}(v + v^* + r),$$

and z approximates x. [Using symbolic matrix notation with respect to $\{p, q, r\}$, $z = \begin{bmatrix} y & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \end{bmatrix}$.]

III.2.2.7. Proposition. A σ-unital purely infinite simple nonunital C*-algebra is stable.

Proof: Suppose A is separable, simple, nonunital, and purely infinite. Then A has a strictly increasing approximate unit (p_n) of projections, and $p_{n+1} - p_n$ is infinite for all n. It is routine using III.2.1.4. to construct a strictly increasing sequence (q_n) of projections such that $p_n \leq q_n \leq p_{n+1}$ and $(q_{n+1} - q_n) \sim q_1$ for all n. The sequence (q_n) is then also an approximate unit for A, and the isomorphism of A with $q_1 A q_1 \otimes \mathbb{K}$ is then obvious. 👍

III.3. Dimension Functions and States

The goal of this section is to prove that if A is simple, unital, and nuclear, then every state on $K_0(A)$ comes from a trace, and that in the weakly unperforated case $K_0(A)$ has the strict ordering from the traces. This discussion is again partially excerpted from [Bla98] and [Bla06].

III.3.1. States on Ordered Groups and Semigroups

The order structure on an ordered group is at least partially (and in good cases completely) determined by the states, which in the K_0 case are closely related to the tracial states on the algebra.

III.3.1.1. Definition. A *state* on a scaled ordered group (G, G_+, u) is an order-preserving homomorphism f from G to $\mathbb{R}$ with $f(u) = 1$.

The set $S(G, G_+, u)$ (or just denoted $S(G)$ when there is no confusion) of all states on (G, G_+, u) is a compact convex set in the topology of pointwise convergence. $S(G)$ is called the *state space* of G.

We now develop some properties of $S(G)$ due to K. Goodearl and D. Handelman [GH76], including a Hahn–Banach type existence theorem.

III.3.1.2. Lemma. Let (G, G_+, u) be a scaled ordered group. Let H be a subgroup of G containing u, and f a state on $(H, H \cap G_+, u)$ (we do not assume H is positively generated). Let $t \in G_+$, and

$$p = \sup\{f(x)/m \mid x \in H, m > 0, x \leq mt\}$$

$$q = \inf\{f(y)/n \mid y \in H, n > 0, nt \leq y\}.$$

Then:

(i) $0 \leq p \leq q < \infty$.

(ii) If g is a state on $(H + \mathbb{Z}t,\, u)$ which extends f, then $p \leq g(t) \leq q$.

(iii) If $p \leq r \leq q$, then there is a unique state g on $(H + \mathbb{Z}t, u)$ which extends f with $g(t) = r$.

The proof is a simple calculation. The next theorem then follows from a Zorn's Lemma argument:

III.3.1.3. Theorem. Let (G, G_+, u) be a scaled ordered group, and let H be a subgroup of G containing u. If f is any state on $(H, H \cap G_+, u)$, then f extends to a state on (G, G_+, u).

III.3.1.4. Corollary. Let (G, G_+, u) be a scaled ordered group, and let $t \in G_+$. Let $f_*(t) = p, f^*(t) = q$ defined as in III.3.1.2. with $H = \mathbb{Z}u$. Then:

(i) $0 \leq f_*(t) \leq f^*(t) < \infty$.

(ii) If f is any state on G, then $f_*(t) \leq f(t) \leq f^*(t)$.

(iii) If $f_*(t) \leq r \leq f^*(t)$, then there is a state g on G with $g(t) = r$.

$f_*(t)$ and $f^*(t)$ can be more elegantly described as

$$f_*(t) = \sup\{n/m \mid nu \leq mt\}$$

$$f^*(t) = \inf\{n/m \mid mt \leq nu\}.$$

III.3.1.5. Theorem. Let (G, G_+, u) be a simple scaled ordered group, and $x \in G$. Then $nx > 0$ for some $n \in \mathbb{N}$ if and only if $f(x) > 0$ for every $f \in S(G)$. If (G, G_+, u) is weakly unperforated, then G has the strict ordering from its states, i.e. $G_+ = \{0\} \cup \{x \mid f(x) > 0 \text{ for all } f \in S(G)\}$.

Proof: If $x > 0$, then x is an order unit, so $u \leq mx$ for some $m > 0$. Then $0 < 1/m \leq f_*(x)$, so $f(x) > 0$ for all $f \in S(G)$. Conversely, suppose $f(x) > 0$ for all $f \in S(G)$. By compactness, we have

$$f_*(x) = \inf_{f \in S(G)} f(x) > 0$$

so there are positive integers n and m with $0 < nu \leq mx$, and therefore $mx > 0$. (Note that this implication does not require G to be simple.)

There is an alternate way to view III.3.1.5.. If $x \in G$, then x induces a continuous affine function $\hat{x}$ on $S(G)$ by $\hat{x}(f) = f(x)$. Thus there is a homomorphism ρ from G to $\text{Aff}(S(G))$, the group of all continuous real-valued affine functions on $S(G)$. III.3.1.5. then says that G has the strict ordering from ρ, in the sense of **??**(ii).

The Semigroup Case

It will turn out not to be adequate to just consider states on ordered groups; we will need analogous results for ordered semigroups. These results are from [BR92] (from which the exposition is excerpted), generalizing [Rør92b]. We have stated the semigroup results in the greatest possible generality, more than is needed for the applications. Extension results for states were established for semigroups equipped with the algebraic preordering by Tarski [Tar38], and for preordered semigroups with cancellation by Aumann [Aum57]. Preordered groups were considered in [Han81]. For our applications we must deal with preordered semigroups which are not algebraically ordered and which do not have cancellation.

III.3.1.6. Definition. A *preordered semigroup* is an abelian semigroup H with a preorder (transitive relation) $<$ which is translation-invariant, i.e. if $x < y$, then $x + z < y + z$ for all $z \in H$. As usual, we write $x \leq y$ if $x < y$ or $x = y$; $\leq$ is then a reflexive transitive relation.

An *order unit* in a preordered semigroup H is an element u such that, for every $x \in H$, there is an $n \in \mathbb{N}$ such that $x \leq x + u$, $x \leq nu$, and $u \leq x + nu$. Note that if u is an order unit and $u \leq v$, then v is also an order unit.

A *scaled preordered semigroup* is a preordered semigroup with distinguished order unit.

A *state* on a scaled preordered semigroup $(H, <, u)$ is an order-preserving homomorphism f from H to the additive group of real numbers $\mathbb{R}$, such that $f(u) = 1$. Denote the set of all states on $(H, <, u)$ by $\mathcal{S}(H, <, u)$, or just by $\mathcal{S}(H)$ when the order structure and order unit are understood. $\mathcal{S}(H)$ is a compact convex set in the topology of pointwise convergence [if $x \leq nu$ and $u \leq x + nu$, then $1 - n \leq f(x) \leq n$ for any $f \in \mathcal{S}(H)$.]

III.3.1.7. Definition. Let ϕ be a homomorphism from a scaled preordered semigroup $(H, <_H, u)$ to a scaled preordered semigroup $(H', <_{H'}, v)$, with $\phi(u) = v$.

ϕ is an *order embedding* if, for all $x, y \in H$, $x <_H y$ if and only if $\phi(x) <_{H'} \phi(y)$.

ϕ is an *approximate order embedding* if, for any $x, y \in H$ with y and $\phi(y)$ order units, there is an $n \in \mathbb{N}$ with $nx + u \leq_H ny$ if and only if there is an $m \in \mathbb{N}$ with $m\phi(x) + v \leq_{H'} m\phi(y)$.

ϕ is a *stable order embedding* if, for any $x, y \in H$, there are $n \in \mathbb{N}$ and $z \in H$ with $nx + z + u \leq_H ny + z$ if and only if there are $m \in \mathbb{N}$ and $w \in H'$ with $m\phi(x) + w + v \leq_{H'} m\phi(y) + w$.

III.3.1.8. It is obvious that an order embedding is an approximate order embedding. It is not *a priori* obvious that an order embedding is a stable order embedding, but this will follow from III.3.1.10. below. Note that if g is a state on H' and $\phi : H \to H'$ is a stable order embedding, then $g \circ \phi$ is a state on H: if $x \leq y$ in H and $k \in \mathbb{N}$, then $kx + z + u \leq_H (ky + u) + x$ for any $z \in H$, and so $mk\phi(x) + w + v \leq_{H'} mk\phi(y) + mv + w$ for some $w \in H'$ and $m \in \mathbb{N}$. Thus $(g \circ \phi)(x) + \frac{1}{mk} \leq_{H'} (g \circ \phi)(y) + \frac{1}{k}$, so $g \circ \phi$ is order-preserving.

III.3.1.9. Lemma. Let $(H, <, u)$ be a scaled preordered semigroup, and $x, y \in H$, with y an order unit. If there is a $z \in H$ such that $x + z + u \leq y + z$, then there is an $n \in \mathbb{N}$ with $nx + u \leq ny$.

Proof: Note that

$$2(x + u) + z = (x + u) + (x + z + u) \leq (x + u) + (y + z) = y + (x + z + u) \leq y + y + z = 2y + z$$

and by induction $mx + mu + z \leq my + z$ for all m. Choose k so that $z \leq ky$, and then choose m so that $kx + u \leq mu + z$. Then

$$(m + k)x + u = mx + kx + u = mx + mu + z \leq my + z \leq my + ky = (m + k)y \ .$$

👍

III.3.1.10. Proposition. Let $(H, <_H, u)$ and $(H', <_{H'}, v)$ be scaled preordered semigroups, and ϕ a homomorphism from H to H' with $\phi(u) = v$. Then ϕ is an approximate order embedding if and only if ϕ is a stable order embedding.

Proof: Suppose ϕ is an approximate order embedding, $x, y \in H$, and $m\phi(x) + w + v \leq_{H'} m\phi(y) + w$ for some $m \in \mathbb{N}$ and $w \in H'$. Choose k large enough that $y + ku$ is an order unit in H and $\phi(y) + kv = \phi(y + ku)$ is an order unit in H'. Then $m\phi(x + ku) + w + v \leq_{H'} m\phi(y + ku) + w$, so by III.3.1.9. there is an $r \in \mathbb{N}$ with $r\phi(x + ku) + v \leq_{H'} r\phi(y + ku)$. Since ϕ is an approximate order embedding, there is an $n \in \mathbb{N}$ with $n(x + ku) + u \leq_H n(y + ku)$. Set $z = nku$. The other half of the proof that ϕ is a stable order embedding is similar.

Now suppose ϕ is a stable order embedding, and $x, y \in H$, with y an order unit, for which there is an $m \in \mathbb{N}$ with $m\phi(x) + v \leq_{H'} m\phi(y)$. Then since ϕ is a stable order embedding there is a $k \in \mathbb{N}$ and $z \in H$ with $kx + z + u \leq_H ky + z$, so by III.3.1.9. there is an $n \in \mathbb{N}$ with$nkx + u \leq_H nky$. The other half of the proof that ϕ is an approximate order embedding is similar. 👍

A result similar to III.3.1.10. for partially ordered groups was established by Handelman [Han95, Lemma 5].

III.3.1.11. Theorem. Let $(H, <_H, u)$ and $(H', <_{H'}, v)$ be scaled preordered semigroups, and let $\phi : H \to H'$ be a homomorphism with $\phi(u) = v$. Then

$$\mathcal{S}(H) = \{g \circ \phi : g \in \mathcal{S}(H')\}$$

if and only if ϕ is a stable order embedding.

III.3.1.12. Corollary. Let $(H, <, u)$ be a scaled preordered semigroup, H_0 a subsemigroup containing u. Give H_0 the relative preordering. Then every state on H_0 extends to a state on H.

Proof: The inclusion map from H_0 into H is clearly an approximate order embedding, hence a stable order embedding by III.3.1.10.. The result then follows from III.3.1.11.. 👍

Theorem III.3.1.11. will be proved using III.3.1.3., although it does not follow immediately. To explain the difficulties, we introduce some notation which will also be used in the proof of III.3.1.11..

We first replace the scaled preordered semigroup $(H, <, u)$ by its maximal partially ordered quotient. Define an equivalence relation $\sim$ on H by $x \sim y$ if $x \leq y$ and $\leq x$, and denote the equivalence class of x by $\langle x \rangle$. Let $\bar{H} = \{\langle x \rangle : x \in H\}$. Define $\langle x \rangle + \langle y \rangle = \langle x + y \rangle$ and $\langle x \rangle \leq \langle y \rangle$ if $x \leq y$; then one obtains a well-defined binary operation making $\bar{H}$ into a partially ordered semigroup. Scale $\bar{H}$ by taking $\langle u \rangle$ as order unit. There is an obvious one-one correspondence between the states on $(H, <, u)$ and the states on $\bar{H}, \leq, \langle u \rangle)$. Going one step further, we may form the Grothendieck group $G(\bar{H})$; $G(\bar{H})$ becomes an ordered group in the sense of () by taking

$$G(\bar{H})_+ = \{[x] - [y] : \langle x \rangle, \langle y \rangle \in \bar{H}, \langle y \rangle \leq \langle x \rangle\}$$

(We denote the image of $\langle x \rangle$ in $G(\bar{H})$ by $[x]$.) The states of $(H, <, u)$ again correspond naturally to the states on $(G(\bar{H}), \leq, [u])$. Note that $[x] \leq [y]$ in $G(\bar{H})$ if and only if there is a $z \in H$ with $x + z \leq y + z$.

Even in the special case of III.3.1.12. when H is partially ordered, the result does not follow immediately from III.3.1.3. for two technical reasons:

(i) The natural map ϕ_G from $G(H_0)$ to $G(H)$ need not be injective in general.

(ii) Even if ϕ_G is injective, it need not be an order-isomorphism onto its image if the order on H is not well behaved.

There is a third technical difficulty in the general case of III.3.1.11., in that ϕ need not drop to a well-defined homomorphism from $G(\bar{H})$ to $G(\bar{H}')$.

We will apply III.3.1.3. in the following form:

III.3.1.13. Lemma. Let $(H, <, u)$ be a scaled preordered semigroup, and $x, y \in H$. Then $f(x) < f(y)$ for every $f \in \mathcal{S}(H)$ if and only if there is an $n \in \mathbb{N}$ and $z \in H$ with $nx + z + u \leq ny + z$.

Proof: It is clear that $nx + z + u \leq ny + z$ for some $z \in H$ and $n \in \mathbb{N}$ implies $f(x) < f(y)$ for all $f \in \mathcal{S}(H)$. Conversely, suppose $f(x) < f(y)$ for all $f \in \mathcal{S}(H)$. Using the bijection between $\mathcal{S}(H)$ and $\mathcal{S}(G(\bar{H}))$, we obtain that $f([y]-[x]) > 0$ for all $f \in \mathcal{S}(G(\bar{H}))$, and it follows from [Goo86, 4.12] that $k([y]-[x])$ is an order unit in $G(\bar{H})$ for some $k \in \mathbb{N}$. So there is an $n \in \mathbb{N}$ with $n([y]-[x]) \geq [u]$, or equivalently $[nx+u] \leq [y]$, and we conclude that $nx + u + z \leq ny + z$ for some $z \in H$.

We will also need the following fact, which follows immediately from [Bla88, 3.4.7] (itself a consequence of III.3.1.3.; see also [Goo86, 7.11]):

III.3.1.14. Lemma. Let $(H, <, u)$ be a scaled preordered semigroup. If K is a closed convex subset of $\mathcal{S}(H)$ such that, for every $x, y \in H$, $f(x) < f(y)$ for all $f \in K$ implies $f(x) - f(y)$ for all $f \in \mathcal{S}(H)$, then $K = \mathcal{S}(H)$.

We now give the proof of Theorem III.3.1.11.:

Proof: Set $K = \{g \circ \phi : g \in \mathcal{S}(H')\}$. If $K = \mathcal{S}(H)$, then it follows immediately from III.3.1.13. that ϕ must be a stable order embedding. Conversely, suppose ϕ is a stable order embedding. Then $K \subseteq \mathcal{S}(H)$, and by III.3.1.13. K satisfies the hypotheses of III.3.1.14., so $K = \mathcal{S}(H)$.

III.3.2. States, Dimension Functions, and Quasitraces

The path to showing that the states on $(K_0(A), K_0(A)_+, [1_A])$ come from traces goes first through dimension functions and then quasitraces.

Dimension Functions

There is a close connection between traces and dimension functions. We specialize to the case of finite dimension functions, which is the relevant case for simple C*-algebras.

III.3.2.1. Definition. A *(finite) dimension function* on a C*-algebra A is a function $d : M_\infty(A)_+ \to [0, \infty)$ such that $d(x) \leq d(y)$ if $x \precsim y$ (II.1.9.8.) and $d(x + y) = d(x) + d(y)$ if $x \perp y$.

Dimension functions measure the "size of the support" of the elements of the algebra.

There is an inconsistency in the literature in the use of the terms *dimension function* and *rank function*. They are sometimes interchangable, or one term or the other is sometimes used for a function which is only defined on projections or is not assumed to extend to matrix algebras. (In general, if it extends to 2×2 matrices, it then extends uniquely to all matrix algebras [BH82].)

III.3.2.2. In general, we also allow a dimension function to take the value $+\infty$. But if d is a dimension function on a C*-algebra A, then it is easily verified that $\{x \in A : d(x^*x) < +\infty\}$ is an ideal of A, so if A is simple and unital, either $d(x) = +\infty$ for all nonzero positive x in A (an uninteresting dimension function we want to exclude) or else $d(x) < +\infty$ for all $x \in A_+$.

III.3.2.3. If d is a (finite) dimension function on a unital C*-algebra A, then $d(x) \leq d(1_A)$ for all $x \in A_+$ (since $x \precsim 1_A$), so a dimension function on a unital C*-algebra is bounded. We say a dimension function d on a unital C*-algebra A is *normalized* if $d(1_A) = 1$ (hence $d(x) \in [0, 1]$ for all $x \in A_+$).

There is a one-one correspondence between normalized dimension functions on A and normalized dimension functions on $M_n(A)$ for any n: a normalized dimension function d on A gives a finite dimension function d_n on $M_n(A)$ with $d_n(1_{M_n(A)}) = n$, so $\frac{1}{n}d_n$ is a normalized dimension function on $M_n(A)$. Conversely, if d is a normalized dimension function on $M_n(A)$, we must have $d(1_A) = \frac{1}{n}$, so nd is a normalized dimension function on A.

III.3.2.4. If A (or $M_n(A)$ for some n) is properly infinite, there are no normalized dimension functions on A: if p and q are projections in A with $p \perp q$ and $p \sim q \sim 1_A$, then for any dimension function d we have

$$d(1_A) + d(1_A) = d(p) + d(q) = d(p + q) \leq d(1_A)$$

so $d(1_A)$ must be either 0 or $+\infty$.

III.3.2.5. If d is a dimension function on A, and $x \in A_+$, then since $x \approx \alpha x$ for all $\alpha > 0$, it follows that $d(\alpha x) = d(x)$ for all $\alpha > 0$. But $d(0) = 0$. Thus d cannot be continuous unless it is identically 0, even if it is finite. A finite dimension function is not even necessarily lower semicontinuous. Lower semicontinuity is a crucial property: dimension functions are analogous to finitely additive measures, and lower semicontinuity corresponds to countable additivity. Fortunately, there are "many" lower semicontinuous dimension functions:

III.3.2.6. Proposition. Let A be a C*-algebra, and d a dimension function on A. Then there is a lower semicontinuous dimension function d' on A such that $d' \leq d$ and $d'(p) = d(p)$ for every projection $p \in M_\infty(A)$. In particular, if A is unital and d is normalized, then d' is also normalized.

Proof: Define $d'(a) = \sup_{\epsilon > 0} d(f_\epsilon(a))$ (). Then $d'(a) \leq d(a)$ for all a, and $d'(p) = d(p)$ for any projection p. If $b \precsim a$, then by II.1.9.22.(v) we have $d'(b) \leq d'(a)$. If $a \perp b$, then $f_\epsilon(a + b) = f_\epsilon(a) + f_\epsilon(b)$ for any $\epsilon > 0$, and so $d'(a + b) = d'(a) + d'(b)$. Finally, d' is lower semicontinuous by II.1.9.20..

III.3.2.6. was first proved in [Han81] for AW*-algebras using a ring-theoretic argument, and the general C*-algebra case was reduced to the AW*-case in [BH82]. This quick proof is from [BR92].

The connection between dimension functions and the states on the Cuntz semigroup is straightforward; in fact, the Cuntz semigroup $W(A)$ (and its Grothendieck group $K_0^*(A)$) were defined precisely to make the following result hold:

III.3.2.7. Proposition. Let A be a unital C*-algebra. There is a natural one-one correspondence between states on the ordered semigroup $(W(A), [1_A])$ and the normalized dimension functions on A. The correspondence also gives a correspondence between the states on the scaled ordered group $(K_0^*(A), [1_A])$ and the dimension functions on A.

Proof: If ϕ is a state on $(W(A), [1_A])$, define a dimension function d_ϕ on A by $d_\phi(x) = \phi([x])$. It is obvious that d_ϕ is a dimension function on A. Conversely, if d is a dimension function on A and hence on $M_\infty(A)$, for $x \in M_\infty(A)$ define $\phi_d([x]) = d(x)$. It is clear that ϕ_d is well defined and order-preserving. If $x, y \in M_\infty(A)$, then

$$\phi_d([x] + [y]) = \phi_d([diag(x, y)]) = d(diag(x, y)) = d(x) + d(y) = \phi_d([x]) + \phi_d([y])$$

so ϕ_d is a homomorphism. And $\phi_d([1_A]) = d(1_A) = 1$, so ϕ_d is a state on $(W(A), [1_A])$. The two maps are obviously inverses. Every state on $(W(A), [1_A])$ is a homomorphism from the semigroup $W(A)$ to the additive group $\mathbb{R}$, hence drops uniquely to a homomorphism from the Grothendieck group $K_0^*(A)$ to $\mathbb{R}$, and every such homomorphism occurs. 👍

Combining this result with previous ones, we obtain the first step in the identification of traces with states on K_0:

III.3.2.8. Theorem. Let A be a simple unital C*-algebra. Then for every state f on $(K_0(A), K_0(A)_+, [1_A])$ there is a lower semicontinuous normalized dimension function d on A such that $f([p]) = d(p)$ for every projection p in $M_\infty(A)$. If $x = [p] - [q] \in K_0(A)$, then $nx > 0$ for some $n \in \mathbb{N}$ if and only if $d(q) < d(p)$ for every lower semicontinuous normalized dimension function d on A. If $(K_0(A), K_0(A)_+, [1_A])$ is weakly unperforated, then it has the strict ordering from the lower semicontinuous normalized dimension functions, i.e. if $0 \neq x = [p] - [q] \in K_0(A)$, then $x \in K_0(A)_+$ if and only if $d(q) < d(p)$ for every lower semicontinuous normalized dimension function d on A.

Proof: If A is not stably finite, then $K_0(A)_+ = K_0(A)$ (III.2.1.6.) and the state space is empty (and there are also no normalized dimension functions on A), so the result holds vacuously. Thus we can assume A is stably finite.

If we can show that every state on $K_0(A)$ comes from a state on $K_0^*(A)$ via the inclusion homomorphism, we are essentially done. This does not follow immediately from III.3.1.3. since the inclusion homomorphism is not injective in general. However, the states on $K_0(A)$ naturally correspond to states on $V(A)$, and the states on $K_0^*(A)$ to states on $W(A)$, and $V(A)$ does embed as a subsemigroup of $W(A)$ in the stably finite case (with the orderings coinciding ()), so we may use the more delicate () to conclude that every state on $V(A)$ extends to a state on $W(A)$, hence comes from a normalized dimension function on A. All the statements of the theorem with "lower semicontinuous" omitted then follow from III.3.2.7.. The "lower semicontinuous" can be added by III.3.2.6. since we are only seeking states on $V(A)$. 👍

III.3.2.9. The first statement of III.3.2.8. also holds for nonsimple C*-algebras: the proof in the properly infinite and stably finite cases is the same, but is more delicate for C*-algebras which are infinite but not properly infinite. See ().

Quasitraces

We now connect dimension functions with quasitraces, a (seemingly) slight generalization of traces. The correspondence can be thought of as a noncommutative extension of the Riesz Representation Theorem.

III.3.2.10. A trace τ on a separable unital C*-algebra A defines a dimension function d_τ. The definition can be made in several ways. Most elegant is to proceed as follows: if $B \cong C(X)$ is a (unital) commutative C*-subalgebra of A, τ defines a state on $C(X)$ and hence a Borel probability measure μ on X. If $f \in C(X)_+$, let

$$d_\tau(f) = \mu(\{x \in X : f(x) \neq 0\}).$$

It is easily checked that this gives a well-defined normalized dimension function on A, which is lower semicontinuous. Explicit formulas for d_τ, which can be used as alternate definitions, are

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}) = \lim_{\epsilon\to 0} \tau(f_\epsilon(a))$$

(II.1.9.16.). The dimension function d_τ is lower semicontinuous ().

III.3.2.11. The procedure can be (almost) reversed. If d is a dimension function on a separable unital C*-algebra A, and $B \cong C(X)$ is a (unital) commutative C*-subalgebra of A, then d defines a finitely additive measure μ on the algebra of subsets of X generated by the open sets, where if U is open, $\mu(U) = d(f)$, where f is any function in $C(X)_+$ nonzero precisely on U. If d is lower semicontinuous, then μ extends to a countably additive Borel measure on X and hence a state (trace) τ on B. The definition of τ on all of A_+ is unambiguous, and τ is a quasitrace:

III.3.2.12. Definition. A *(finite) quasitrace* on a C*-algebra A is a function $\tau : A_+ \to [0, \infty)$ such that $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$, $\tau(\lambda a) = \lambda\tau(a)$ for all $\lambda \geq 0$ and $a \in A_+$, and $\tau(a + b) = \tau(a) + \tau(b)$ if a and b are commuting elements of A_+, and such that τ extends to a map $M_n(A)_+ \to [0, \infty)$ with the same properties. τ is *normalized* if $\sup\{\tau(a) : 0 \leq a \leq 1\} = 1$ (equivalently, if A is unital, $\tau(1_A) = 1$). Denote by $QT(A)$ the set of normalized quasitraces on A. $QT(A)$ is a convex set, which is compact in the topology of elementwise convergence if A is unital.

As with dimension functions, we should also allow τ to take the value $+\infty$; but $\{x \in A : \tau(x^*x) < +\infty\}$ is an ideal of A, so if A is simple and unital either τ is finite everywhere and hence bounded, or $\tau(x) = +\infty$ for all nonzero $x \in A_+$, an uninteresting case we want to exclude.

A (finite) quasitrace τ can be extended in a well-defined way to all elements: since it is linear on commutative C*-subalgebras, it is well defined on self-adjoint (and, more generally, normal) elements, and we can define $\tau(x + iy) = \tau(x) + i\tau(y)$ for x, y self-adjoint. The quasitrace τ is a trace if and only if this extension is linear on all of A (or, equivalently, if τ is additive on A_+).

III.3.2.13. A C*-algebra A with a separating family of tracial states (or, more generally, normalized quasitraces (III.3.2.12.)) must be finite: if $x \in A$ with $x^*x = 1_A$, then $\tau(1_A - xx^*) = 0$ for every normalized quasitrace τ. Since $M_n(A)$ also has a separating family of normalized quasitraces, A must be stably finite.

III.3.2.14. A normalized quasitrace τ on a C*-algebra A defines a normalized dimension function d_τ on A exactly as in III.3.2.10. (linearity is only needed on commutative C*-subalgebras). The constructions of III.3.2.10. and III.3.2.11. are mutual inverses, so we obtain:

III.3.2.15. Proposition. Let A be a C*-algebra. There is a natural one-one correspondence $\tau \mapsto d_\tau$ between the set $QT(A)$ of normalized quasitraces on A and the set of lower semicontinuous normalized dimension functions on A. If p is a projection in $M_\infty(A)$, then $d_\tau(p) = \tau(p)$.

Thus we "almost" obtain our final result from III.3.2.8.:

III.3.2.16. Theorem. Let A be a simple unital C*-algebra. Then for every state f on $(K_0(A), K_0(A)_+, [1_A])$ there is a normalized quasitrace τ on A such that $f([p]) = \tau(p)$ for every projection p in $M_\infty(A)$. If $x = [p] - [q] \in K_0(A)$, then $nx > 0$ for some $n \in \mathbb{N}$ if and only if $\tau(q) < \tau(p)$ for every normalized quasitrace τ on A. If $(K_0(A), K_0(A)_+, [1_A])$ is weakly unperforated, then it has the strict ordering from the normalized quasitraces, i.e. if $0 \neq x = [p] - [q] \in K_0(A)$, then $x \in K_0(A)_+$ if and only if $\tau(q) < \tau(p)$ for every normalized quasitrace τ on A.

The first statement again holds also for nonsimple C*-algebras.

III.3.2.17. The correspondence described above gives a continuous affine map

$$\theta_A : QT(A) \to S = S(K_0(A)).$$

By III.3.2.16., the map θ_A is always surjective. In other words, if A is any unital C*-algebra, then every state on $K_0(A)$ comes from a normalized quasitrace on A.

This map is not injective in general. For example, if $A = C([0,1])$, then $QT(A) = \mathcal{T}(A)$ is the state space of A, while S is a singleton. S is not even a simplex in general (I.3.7.14.). The map θ_A is injective if A has real rank zero (**??**).

If A is a stably finite unital C*-algebra, then $(K_0(A), K_0(A)_+, [1_A])$ has at least one state (). Thus we obtain another important conclusion:

III.3.2.18. Corollary. Let A be a stably finite unital C*-algebra. Then A has at least one normalized quasitrace.

Quasitraces and Traces

III.3.2.19. Every trace is a (normalized) quasitrace. One of the oldest, most famous, and most important open general structure questions for C*-algebras is whether every quasitrace is a trace (I.1.6.38.). The general question can be reduced to the case of AW*-factors [BH82]. U. Haagerup showed that the answer is yes for well-behaved C*-algebras:

III.3.2.20. Theorem. Every quasitrace on an exact C*-algebra (**??**, **??**) is a trace.

The first proof of this theorem was in [Haa14], which appeared in handwritten form in 1991 but was not published until 2014; see [HT99] for a proof using random matrices. Actually, these references only treat the case of normalized quasitraces on unital C*-algebras; E. Kirchberg (cf. [**?**]) observed that the general case follows from this. See () for a proof in the case of finite nuclear dimension.

Combining this result with III.3.2.16., we obtain:

III.3.2.21. Corollary. Let A be a stably finite unital exact C*-algebra. Then A has a trace. If A is simple, the traces on A determine the ordering on $K_0(A)$ up to perforation; if $K_0(A)$ is weakly unperforated, then $K_0(A)$ has the strict ordering from $\mathcal{T}(A)$, i.e. a nonzero element x of $K_0(A)$ is in $K_0(A)_+$ if and only if $[\theta_A(\tau)](x) > 0$ for all $\tau \in \mathcal{T}(A)$.

III.3.2.22. Let $\mathcal{T}(A)$ and $QT(A)$ denote respectively the traces and normalized quasitraces on A. We have $\mathcal{T}(A) \subseteq QT(A)$. It is at least true that $QT(A)$, like $\mathcal{T}(A)$, is always a Choquet simplex, which is metrizable if A is separable [BH82, II.4.4]. For the ordering on $K_0(A)$ the set $QT(A)$ is the more natural and important set to consider, due to the fact that the quasitraces on A are in one-one correspondence with the lower semicontinuous dimension functions on A (III.3.2.11.). $\mathcal{T}(A)$ is a face in $QT(A)$.

III.3.3. Quasitraces and Traces on Nuclear C*-Algebras

References: [Cun78], [GH76], [BH82], [BR92], [BW11]

III.4. Definition and Description of the Elliott Invariant (Unital Case)

We recapitulate the properties of the Elliott Invariant, beginning with a careful definition. While the invariant can be defined more generally, we stick to the case of separable simple unital nuclear C*-algebras in describing its properties.

III.4.1. Definition of the Elliott Invariant

III.4.1.1. Definition. Let A be a unital C*-algebra. The *Elliott invariant* $Ell(A)$ of A is the 6-tuple

$$Ell(A) = (K_0(A), K_0(A)_+, [1_A], \mathcal{T}(A), \langle\cdot,\cdot\rangle_A; K_1(A))$$

where

$K_0(A)$ and $K_1(A)$ are the K-groups of A (I.3.3.10., I.3.6.4.).

$K_0(A)_+$ is the positive cone of $K_0(A)$, the image of $V(A)$ in $K_0(A)$ (I.3.3.12.).

$[1_A]$ is the class of the unit of A in $K_0(A)_+$ (I.3.3.15.).

$\mathcal{T}(A)$ is the set of traces (tracial states) of A (I.1.6.1.).

$\langle\cdot,\cdot\rangle_A$ is the pairing of $\mathcal{T}(A)$ and $K_0(A)$ (I.3.3.23.).

III.4.1.2. The pairing $\langle \cdot, \cdot \rangle_A$ is the map from $\mathcal{T}(A) \times K_0(A)$ to $\mathbb{R}$ given by

$$\langle \tau, [p] - [q] \rangle_A = \tau_\infty(p) - \tau_\infty(q)$$

where p and q are projections in $M_\infty(A)$ (I.1.7.12.) and τ_∞ is the canonical extension of τ to $M_\infty(A)$. This pairing is well defined by (), and gives a pairing of $\mathcal{T}(A)$ with all of $K_0(A)$ since every element of $K_0(A)$ is of the form $[p] - [q]$ for projections $p, q \in M_\infty(A)$. The pairing can be described in two alternate ways:

The homomorphism $\rho_A : K_0(A) \to \text{Aff}(\mathcal{T}(A))$ (I.3.3.57.); $\rho_A(x)$ is often written $\hat{x}$, i.e. $\hat{x}(\tau) = [\rho_A(x)](\tau) = \langle \tau, x \rangle_A$.

The affine map $\theta_A : \mathcal{T}(A) \to \mathcal{S}(K_0(A), K_0(A)_+, [1_A])$ (I.3.3.23.).

III.4.2. Properties of the Elliott Invariant

III.4.2.1. Let A be a separable simple unital nuclear C*-algebra. Then the Elliott invariant $Ell(A)$ has the following properties:

$K_0(A)$ and $K_1(A)$ are countable abelian groups (I.3.3.11., I.3.6.4., I.3.6.7.).

$K_0(A)_+$ is a subsemigroup of $K_0(A)$ containing 0, which generates $K_0(A)$ as a group.

$(K_0(A), K_0(A)_+)$ is a simple preordered group, i.e. for any $u \in K_0(A)_+$ and $x \in K_0(A)$ there is an $n \in \mathbb{N}$ with $nu - x \in K_0(A)_+$ (). In particular, $[1_A]$ is an order unit in $(K_0(A), K_0(A)_+)$.

If A is not stably finite, then

$K_0(A)_+ = K_0(A)$ (III.2.1.6.).

$\mathcal{T}(A) = \emptyset$ (I.1.6.37.) (and thus the pairing $\langle \cdot, \cdot \rangle_A$ is vacuous).

If A is stably finite, then

$\mathcal{T}(A) \neq \emptyset$ and is a metrizable Choquet simplex (III.3.2.21., I.3.3.55.).

$K_0(A)_+ \cap [-K_0(A)_+] = \{0\}$ (), i.e. $(K_0(A), K_0(A)_+)$ is a simple ordered group.

The affine map $\theta_A : \mathcal{T}(A) \to \mathcal{S}(K_0(A), K_0(A)_+, [1_A])$ is surjective (III.3.2.16., III.3.2.20.).

Up to perforation, the ordering on $K_0(A)$ is the strict ordering from the pairing with $\mathcal{T}(A)$, i.e. if $x \in K_0(A)$, there is an $n \in \mathbb{N}$ with $nx > 0$ (i.e. $nx \in K_0(A)_+ \setminus \{0\}$) if and only if $\langle \tau, x \rangle > 0$ for all $\tau \in \mathcal{T}(A)$ (III.3.2.16., III.3.2.21.).

III.4.2.2. The Elliott invariant is functorial: if A and B are unital C*-algebras and $\phi : A \to B$ is a unital homomorphism, then

ϕ induces homomorphisms $K_0(\phi) : K_0(A) \to K_0(B)$ and $K_1(\phi) : K_1(A) \to K_1(B)$.

$K_0(\phi)$ maps $K_0(A)_+$ into $K_0(B)_+$ and $[K_0(\phi)]([1_A]) = [1_B]$, i.e. $K_0(\phi)$ is a homomorphism of scaled ordered groups.

ϕ induces an affine map $T_\phi : \mathcal{T}(B) \to \mathcal{T}(A)$ by $T_\phi(\tau) = \tau \circ \phi$, and hence a scaled ordered linear map Θ_ϕ from $\text{Aff}(\mathcal{T}(A))$ to $\text{Aff}(\mathcal{T}(B))$.

$\langle T_\phi(\tau), x \rangle_A = \langle \tau, [K_0(\phi)](x) \rangle_B$ for all $\tau \in \mathcal{T}(B)$ and $x \in K_0(A)$, i.e. $\theta_B \circ K_0(\phi) = \Theta_\phi \circ \theta_A$.

III.5. The Elliott Invariant in the Nonunital Case

There are some subtleties in extending even the definition of the Elliott invariant to the nonunital case, discussed in this section. The results here reduce to the previous ones for unital algebras.

III.5.1. K-Theory of Nonunital C*-Algebras

III.5.2. The Trace Cone

III.5.3. Pairing with K_0

III.5.4. Properties of the Elliott Invariant (Nonunital Case)

III.6. Strict Comparability and the Cuntz Semigroup

III.6.1. Comparability and Order on Positive Elements

III.6.2. Structure of the Cuntz Semigroup

This discussion of the structure of the Cuntz semigroup is valid only for *simple* C*-algebras. The structure is far more complicated in the nonsimple case since the ideal structure is encoded in the Cuntz semigroup. However, separability will not be needed for the general discussion.

Some Important Subsemigroups

III.6.2.1. Recall the definition of the Cuntz semigroup in (). If A is a simple C*-algebra, the Cuntz semigroup $\mathrm{Cu}(A)$ has three important subsemigroups:

> The image of $V(A)$ (), the classes of projections (or well-supported elements) in $A \otimes \mathbb{K}$. If A is stably finite (the important case for the Cuntz semigroup), this subsemigroup is isomorphic to $V(A)$, so by slight abuse we will call it $V(A)$ (we can eliminate the abuse in the infinite case; cf. ()).
>
> The set $W_0(A)$, the classes of (positive) elements of $Ped(A \otimes \mathbb{K})$ which are not well-supported.
>
> The set $\mathrm{Cu}_\infty(A)$, the remaining elements of $\mathrm{Cu}(A)$, i.e. the classes of (positive) elements of $A \otimes \mathbb{K}$ which are not in $Ped(A \otimes \mathbb{K})$; cf. ().

Some of these subsemigroups can be empty. For example, $W_0(\mathbb{C}) = \emptyset$ and $\mathrm{Cu}_\infty(A) = \emptyset$ if A is purely infinite. $V(A)$ is always nonempty since it contains $[0]$, which is the unit of $\mathrm{Cu}(A)$, but it can just be $\{[0]\}$. If A is nonelementary, then $Ped(A)$ contains a positive element with infinite spectrum, hence by functional calculus a positive element which is not well-supported, so $W_0(A)$ is nonempty.

III.6.2.2. Each of these subsemigroups absorbs the previous ones:

> $V(A) + W_0(A) \subseteq W_0(A)$.
>
> $W_0(A) + \mathrm{Cu}_\infty(A) \subseteq \mathrm{Cu}_\infty(A)$.
>
> $V(A) + \mathrm{Cu}_\infty(A) \subseteq \mathrm{Cu}_\infty(A)$.

The semigroup $W(A)$ is the (disjoint) union of $V(A)$ and $W_0(A)$.

Ordering on the Cuntz Semigroup

The Cuntz semigroup is a preordered semigroup (III.3.1.6.) which is partially ordered in the stably finite case but not in general.

III.6.2.3. Definition. Let S be an abelian semigroup. The *algebraic preordering* on S is the preordering defined by $x \leq_a y$ if there is a $z \in S$ with $x + z = y$. If the algebraic preordering is a (partial) ordering, it is called the *algebraic ordering*.

III.6.2.4. The algebraic preordering on a semigroup S is reflexive if S has a unit (additive identity). If $(S, \leq)$ is a preordered semigroup with the property that $x \leq x + y$ for all $x, y \in S$, and $\leq_a$ is the algebraic preordering on S, then $\leq_a$ is weaker than $\leq$, i.e. $x \leq_a y$ implies $x \leq y$.

III.6.2.5. If A is stably finite, the restriction of the ordering on $\mathrm{Cu}(A)$ to $V(A)$ is the usual ordering on $V(A)$, which is the algebraic ordering. However, the restriction to $W_0(A)$ is not the algebraic ordering in general:

III.6.2.6. Example. Let A be the simple unital AF algebra with dimension group $\mathbb{Q}^2$ with strict ordering and order unit $(1,1)$. Let (p_n) and (q_n) $(n > 2)$ be increasing sequences of projections in A with

$$[p_n] = \left(1 - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}\right)$$

$$[q_n] = \left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{2n}\right)$$

and $q_n \leq p_n$ for all n. Set $a = \sum_{n=3}^{\infty} 2^{-n} p_n$ and $b = \sum_{n=3}^{\infty} 2^{-n} q_n$. Then $b \leq a$ in A_+, so $[b] \leq [a]$ in $W_0(A)$. But there is no c with $[b] + [c] = [a]$, so $[b] \not\leq_a [a]$ in $W_0(A)$ (or in $\mathrm{Cu}(A)$). In fact in this case $W_0(A)$ is isomorphic to the open first quadrant in $\mathbb{R}^2$ with the *ordinary* ordering, not the algebraic ordering which is the strict ordering.

$\tilde{\mathcal{T}}(A)$ and its Topology

III.6.2.7. Recall the definition of $\tilde{\mathcal{T}}(A)$ for a (simple) C*-algebra A (). We put a topology on $\tilde{\mathcal{T}}(A)$ as follows.

III.6.2.8. Definition. Let A be a simple C*-algebra. Let $\mathrm{Laff}(A)$ be the set of functions $f : \mathcal{T}(A \otimes \mathbb{K}) \to [0, +\infty]$ such that

$f(0) = 0$.

f is either identically 0, or strictly positive in the sense that if $\tau \in \tilde{\mathcal{T}}(A \otimes \mathbb{K})$, $\tau \neq 0$, then $f(\tau) > 0$.

f is affine.

f is homogeneous, i.e. if $\tau \in \tilde{\mathcal{T}}(A \otimes \mathbb{K})$ and $\lambda \in (0, +\infty)$, then $f(\lambda\tau) = \lambda f(\tau)$.

f is lower semicontinuous for the topology on $\tilde{\mathcal{T}}(A \otimes \mathbb{K})$ defined in ().

Note that Laff(A) is an additive semigroup under the usual pointwise addition of functions, and a cone. The semigroup Laff(A) has cancellation.

Let Laff$_f(A)$ be the set of f in Laff(A) not taking the value $+\infty$ anywhere, and Laff$_\infty(A)$ the complement, i.e. the set of f such that $f(\tau) = +\infty$ for at least one τ. Then Laff$_f(A)$ and Laff$_\infty(A)$ are subsemigroups of Laff(A), and Laff$_f(A)$ + Laff$_\infty(A) \subseteq$ Laff$_\infty(A)$.

Give Laff(A) the usual ordering on functions: $f \leq g$ iff $f(\tau) \leq g(\tau)$ for all $\tau \in \tilde{\mathcal{T}}(A \otimes \mathbb{K})$. With this ordering, Laff(A) becomes an ordered semigroup.

We should properly write Laff$(\tilde{\mathcal{T}}(A \otimes \mathbb{K}))$ instead of Laff(A), but we have chosen economy of terminology. Note that if A and B are stably isomorphic, then Laff(A) and Laff(B) are isomorphic.

III.6.2.9. If A is unital, then there is a natural identification of the strict positive cone of Aff$(\mathcal{T}(A))$ with the continuous functions in Laff$_f(A)$.

The Representation Map

III.6.2.10. There is a well-defined map $\tilde{\rho}_A : \mathrm{Cu}(A) \to \mathrm{Laff}(A)$ defined for $a \in (A \otimes \mathbb{K})_+$ by

$$\tilde{\rho}_A([a])(\tau) = D_\tau(a)$$

where D_τ is the (lower semicontinuous) dimension function associated with τ (), i.e.

$$D_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}) = \sup_{\epsilon>0} \tau(f_\epsilon(a)) \ .$$

This map is an order-preserving semigroup homomorphism.

The next result is (essentially) the main point of the construction of the Cuntz semigroup in [Cun78]:

III.6.2.11. Proposition. Let A be a C*-algebra. Every homomorphism from Cu(A) to $[0, +\infty]$ is given by a lower semicontinuous dimension function.

Combining this with (), we obtain:

Thus in a certain sense $\tilde{\rho}_A$ determines the ordering on Cu(A).

III.6.2.12. Definition. Let A be a simple C*-algebra. A has *smooth Cuntz semigroup* if

(i) If $x, y \in \mathrm{Cu}(A)$ and $[\rho_A(x)](\tau) < [\tilde{\rho}_A(y)](\tau)$ for all $\tau \in \tilde{\mathcal{T}}(A \otimes \mathbb{K})$, then $x \leq y$.

(ii) If $x \in W_0(A)$, $y \in \mathrm{Cu}(A)$, and $\tilde{\rho}_A(x) \leq \tilde{\rho}_A(y)$, then $x \leq y$.

(iii) $V(A)$ has cancellation.

The Real Rank Zero Case

The Scale

So far none of the construction distinguishes between stably isomorphic algebras. But as with K_0, we can make a distinction using a notion of scale.

III.6.2.13. Proposition. Let A be a σ-unital (e.g. separable) C*-algebra (not necessarily simple). Then any two strictly positive elements of A are Cuntz equivalent. In fact, if h is a strictly positive element of A, then $a \precsim h$ for all $a \in A_+$.

In light of this result, the following definition makes sense:

III.6.2.14. Definition. Let A be a σ-unital C*-algebra. The *scale* $sc(A)$ of A in $\mathrm{Cu}(A)$ is the equivalence class of the strictly positive elements of A.

If A is unital, then $sc(A)$ is the class of 1_A, so is in $V(A)$ and coincides with the definition of scale in (). If A is nonunital, then $sc(A)$ is in $W_0(A) \cup \mathrm{Cu}_\infty(A)$. If $x = [b] \in \mathrm{Cu}(A)$, $b \neq 0$, then x is the scale of a C*-algebra stably isomorphic to A (if A is simple), namely the hereditary C*-subalgebra $(A \otimes \mathbb{K})_b$ of $A \otimes \mathbb{K}$. We have $0 \leq [a] \leq sc(A)$ for all $a \in A_+$.

III.6.2.15. We also define the *scale set* $Sc(A)$ to be the set of elements of $\mathrm{Cu}(A)$ which are equivalence classes of (positive) elements of A. This set is defined even if A is not σ-unital, and is an upward directed subset of $\mathrm{Cu}(A)$ since if $a, b \in A_+$, then $[a], [b] \leq [a + b]$ (it is not clearly hereditary in general). If A is σ-unital, then $Sc(A)$ has a largest element, namely $sc(A)$.

III.7. The Cuntz Semigroup

In this section, we more carefully analyze the structure of the Cuntz semigroup of a C*-algebra.

III.7.1. Basic Definitions

Recall the definition () of Cuntz comparison and equivalence:

III.7.1.1. Definition. Let A be a (local) C*-algebra, $a, b \in A$. Then $b \precsim a$ (in A) if there are sequences (x_n), (y_n) in A with $x_n a y_n \to b$. Equivalently, $b \precsim a$ if for every $\epsilon > 0$ there are elements $x, y \in A$ with $\|xay - b\| < \epsilon$. We say a and b are *Cuntz equivalent* (in A), written $a \approx b$, if $a \precsim b$ and $b \precsim a$.

III.7.1.2. A technical difficulty with Cuntz comparison is that if $b \precsim a$, i.e. there are sequences (x_n), (y_n) with $x_n a y_n \to b$, the sequences (x_n) and (y_n) are not necessarily uniformly bounded (and cannot be chosen uniformly bounded in general). Thus it is not entirely obvious that $\precsim$ is transitive. But if $c \precsim b$ and $b \precsim a$, and $\epsilon > 0$, there are $x, y \in A$ with $\|c - xby\| < \frac{\epsilon}{2}$, and then $z, w \in A$ with $\|b - zaw\| < \frac{\epsilon}{2\|x\|\|y\|}$; then

$$\|c - xzawy\| \leq \|c - xby\| + \|xby - xzawy\| = \|c - xby\| + \|x(b - zaw)y\| < \frac{\epsilon}{2} + \|x\|\|y\|\frac{\epsilon}{2\|x\|\|y\|} = \epsilon$$

so $c \precsim a$. Thus $\precsim$ is reflexive and transitive, and $\approx$ is an equivalence relation. Note that these relations are dependent on the containing C*-algebra A; if necessary, we will write $\precsim_A$ and $\approx_A$. In many references, $\approx$ is written $\sim$, but this conflicts with Murray-von Neumann equivalence in some cases (III.7.1.10.).

Cuntz comparison measures the "width" of elements and not the "height." We have:

III.7.1.3. Proposition. Let A be a local C*-algebra. Then

(i) If $a \in A$ and $0 \neq \lambda \in \mathbb{C}$, then $a \approx \lambda a \approx a^* \approx a^*a \approx aa^*$.

(ii) If $a, b \in A$, then $ab \precsim a$ and $ba \precsim a$.

(iii) If $a, b \in A$ and $\overline{b^*Ab} \subseteq \overline{a^*Aa}$, then $b \precsim a$. In particular, if $\overline{b^*Ab} = \overline{a^*Aa}$, then $b \approx a$.

(iv) If $a \in A_+$, then $a \approx a^\alpha$ for any $\alpha > 0$.

(v) If $a, b \in A$, and $b^*b \leq a^*a$, then $b \precsim a$. In particular, if $a, b \in A_+$ and $b \leq a$, then $b \precsim a$.

(vi) If $a, b \in A_+$, then $b \precsim a$ if and only if there is a sequence (x_n) in A with $x_n^* a x_n \to b$, i.e. for any $\epsilon > 0$ there is an $x \in A$ with $\|x^*ax - b\| < \epsilon$.

Proof: (ii): Obvious if A is unital. In general, let (h_n) be an approximate unit for $\overline{a^*Aa}$; then $h_n ab \to ab$ and $bah_n \to ba$. Similarly, if $a \in A$ and $\lambda \in \mathbb{C}$, then $(\lambda h_n)ah_n \to \lambda a$, so $\lambda a \precsim a$; if $\lambda \neq 0$ we also have $a = \lambda^{-1}(\lambda a) \precsim \lambda a$, so $a \approx \lambda a$, part of (i).

(iv): If $\alpha > 1$, we have $a^\alpha = aa^{\alpha-1}$, so $a^\alpha \precsim a$ by (ii). For the opposite inequality, it suffices to show $a^{1/2} \precsim a$. For $n \in \mathbb{N}$ let $x_n = f_{1/n}(a)$ (). Then by functional calculus there is an element $y_n \in A$ with $y_n a x_n = a^{1/2} x_n$ [$y_n = g(a)$, where $g(t) = t^{-1/2}$ for $t \geq 1/2n$ and $g(0) = 0$]. Since $a^{1/2} x_n \to a^{1/2}$, the result follows.

(i): If $a \in A$, we have $a^*a \precsim a$ by (ii). Conversely, $a = x(a^*a)^{1/4}$ for some $x \in A$ by (), so $a \precsim (a^*a)^{1/4} \approx a^*a$ by (iv). Similarly, $a = (aa^*)^{1/4}y$ for some $y \in A$, so $aa^* \precsim a \precsim (aa^*)^{1/4} \approx aa^*$. Thus $a \approx a^*$ by transitivity.

(v): If $b^*b \leq a^*a$, then there is an $x \in A$ with $b = x(a^*a)^{1/4}$ by (). Thus $b \precsim (a^*a)^{1/4} \approx a^*a \approx a$. For the last statement, apply this to $b^{1/2}$ and $a^{1/2}$ to obtain $b \approx b^{1/2} \precsim a^{1/2} \approx a$.

(vi): If there is a sequence (x_n) with $x_n^*ax_n \to b$, then $b \precsim a$. Conversely, suppose $b \precsim a$, so $b^{1/2} \precsim a^{1/2}$, i.e. there are sequences $(x_n), (y_n)$ with $x_n a^{1/2} y_n \to b^{1/2}$. Then

$$y_n^* a^{1/2} x_n^* x_n a^{1/2} y_n \to b \ .$$

If $c_n = a^{1/2}x_n^*x_na^{1/2}$, then $c_n \leq \|x_n\|^2 a$, so by () there is a z_n such that $c_n = z_n^* a^{1/2} z_n$. Then $y_n^* z_n^* a^{1/2} z_n y_n \to b$. So if $\epsilon > 0$, there is a $w \in A$ with $\|w^*a^{1/2}w - b\| < \frac{\epsilon}{2}$. There is a $v \in A$ such that

$$\|v^*av - a^{1/2}\| < \frac{\epsilon}{2\|w\|^2}$$

$$\|w^*v^*avw - b\| \leq \|w^*v^*avw - w^*a^{1/2}w\| + \|w^*a^{1/2}w - b\| < \|w\|^2 \frac{\epsilon}{2\|w\|^2} + \frac{\epsilon}{2} = \epsilon \ .$$

(iii): Replacing b by b^*b, we may assume $b \geq 0$. Then since $b^{1/2} \in \overline{a^*Aa}$ and $((a^*a)^{1/n})$ is an approximate unit for $\overline{a^*Aa}$, we have $b^{1/2}(a^*a)^{1/n}b^{1/2} \to b$. So if $\epsilon > 0$, there is an n such that $\|b^{1/2}(a^*a)^{1/n}b^{1/2} - b\| < \frac{\epsilon}{2}$. For this n, by functional calculus there is an $x \in A$ (even positive and commuting with a^*a) such that $\|x(a^*a) - (a^*a)^{1/n}\| < \frac{\epsilon}{2\|b\|}$. Then

$$\|b^{1/2}xa^*ab^{1/2} - b\| \leq \|b^{1/2}xa^*ab^{1/2} - b^{1/2}(a^*a)^{1/n}b^{1/2}\| + \|b^{1/2}(a^*a)^{1/n}b^{1/2} - b\| < \|b\|\frac{\epsilon}{2\|b\|} + \frac{\epsilon}{2} = \epsilon \ .$$

Thus $b \precsim a$.

III.7.1.4. **Remark:** In connection with (v), note that $0 \leq b \leq a$ does not imply $b^2 \leq a^2$ in A in general unless a and b commute. But for general positive a and b, $b \precsim a$, we have $b^2 \approx b \precsim a \approx a^2$.

III.7.1.5. Since $a \approx a^*a$ for any a, one can just work with positive elements of A, in which case the ordering is as given in I.6.4.5.(vi). One should think of $b \precsim a$ as meaning the support of b is "smaller" ("not larger") than the support of a, although not necessarily in the same "location," and similarly for $a \approx b$.

The following technical fact from [Rør92a] is useful. Recall the functional calculus definitions of $f_\epsilon(a)$ and $(a-\epsilon)_+$ for $a \geq 0$ and $\epsilon > 0$ (). If $a, b \in A_+$, write $b \preceq a$ if there is a $z \in A$ with $b = z^*az$, and $b \precsim\!\!\!\!\approx a$ if there is a $z \in A$ with $b \leq z^*az$. Note that $b \preceq a \Rightarrow b \precsim\!\!\!\!\approx a \Rightarrow b \precsim a$.

III.7.1.6. Lemma. Let A be a C*-algebra, $c, d \in A_+$. If $\|c - d\| < \epsilon < \frac{1}{2}$, then $f_\epsilon(c) \precsim\!\!\!\!\approx d$.

Proof: Let $\eta = \|c - d\| < \epsilon$. Then $c - \eta 1 \leq d$ (in $A^\dagger$). Then

$$(\epsilon - \eta)f_\epsilon(c) \leq f_\epsilon(c)^{1/2}(c - \eta 1)f_\epsilon(c)^{1/2} \leq f_\epsilon(c)^{1/2} d f_\epsilon(c)^{1/2}$$

so $f_\epsilon(c) \leq x^*dx$ for $x = (\epsilon - \eta)^{-1/2} f_\epsilon(c)^{1/2}$.

III.7.1.7. Proposition. Let A be a C*-algebra and $a, b \in A_+$. The following are equivalent:

(i) $b \precsim a$.

(ii) $f_\epsilon(b) \precsim a$ for all $\epsilon > 0$.

(ii′) $(b-\epsilon)_+ \precsim a$ for all $\epsilon > 0$.

(iii) $f_\epsilon(b) \preceq a$ for all $\epsilon > 0$.

(iii′) $(b-\epsilon)_+ \preceq a$ for all $\epsilon > 0$.

(iv) For every $\epsilon > 0$ there is a $\delta > 0$ such that $f_\epsilon(b) \preceq f_\delta(a)$.

(iv′) For every $\epsilon > 0$ there is a $\delta > 0$ such that $(b-\epsilon)_+ \preceq (a-\delta)_+$.

Proof: Since by functional calculus $(c-\epsilon)_+ \preceq f_\epsilon(c) \preceq (c-\epsilon)_+ \preceq c$ for any $c \geq 0$ and $\epsilon > 0$, (ii) $\iff$ (ii′), (iii) $\iff$ (iii′), and (iv) $\iff$ (iv′), and (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (ii).

(iii′) $\Rightarrow$ (i): For each n choose z_n so that $\left(b - \frac{1}{n}\right)_+ = z_n^* a z_n$. Then $z_n^* a z_n \to b$, so $b \precsim a$.

(i) $\Rightarrow$ (iv): Let $\epsilon > 0$, and $x \in A$ with $\|b - x^*ax\} < \epsilon$. Then there is a $\delta > 0$ such that $\|b - x^*(a-2\delta)_+x\| < \epsilon$. Thus there is a $y \in A$ with $\|b - y^* f_{2\delta}(a) y\| < \epsilon$. So by Lemma III.7.1.6. there is a $z \in A$ with $f_\epsilon(b) \leq z^* f_{2\delta}(a) z$. Set $w = z^* f_{2\delta}(a)^{1/2}$, and let $w = u(w^*w)^{1/2}$ be the polar decomposition in A^{**}. Then $s = u(w^*w)^{1/4} \in A$ [Bla06, III.5.2.16], and $sf_\delta(a) = s$, so

$$(sf_\delta(a)s^*)^2 = (ss^*)^2 = ww^* = z^* f_{2\delta}(a) z \geq f_\epsilon(b)\ .$$

thus, by [Bla06, II.3.2.1], $f_\epsilon(b) = r^* f_\delta(a) r$ for some $r \in A$.

(ii) $\Rightarrow$ (iv) by the same argument applied to $f_{\epsilon/2}(a)$ in place of a. 👍

III.7.1.8. Because of the technical difficulty with Cuntz comparison that if $b \precsim a$, i.e. there are sequences (x_n), (y_n) with $x_n a y_n \to b$, the sequences (x_n) and (y_n) are not necessarily uniformly bounded (even if a and b are positive, so there is a sequence (x_n) with $x_n^* a x_n \to b$, the sequence (x_n) cannot generally be chosen uniformly bounded, e.g. if $b = a^{1/2}$), the next proposition, which would be almost trivial if the sequences were uniformly bounded, is a bit delicate to prove (the assertion in [Bla06, II.3.4.7] that this is "straightforward" is too cavalier). But it follows easily from III.7.1.6. and III.7.1.7.:

III.7.1.9. Proposition. Let A be a C*-algebra, $a, b, b_n \in A$. If $b_n \precsim a$ for all n and $b_n \to b$, then $b \precsim a$.

Proof: We have $b_n^* b_n \to b^* b$, so we may assume $b_n, b \geq 0$. Let $\epsilon > 0$. Fix n with $\|b_n - b\| < \epsilon$. Then $f_\epsilon(b) \precapprox b_n \precsim a$ by III.7.1.6.. Thus by III.7.1.7. $b \precsim a$. 👍

In fact, if $a_n \to a$ in A_+, then for any $0 < \delta < \epsilon$ we have $f_\epsilon(a) \precapprox f_\delta(a_n)$ and $f_\epsilon(a_n) \precapprox f_\delta(a)$ for all sufficiently large n [?].

Projections and Well-Supported Elements

III.7.1.10. If p and q are projections, it is easy to check () that $p \precsim q$ in the sense of III.7.1.1. if and only if $p \precsim q$ in the sense of I.1.6.24., so the notation is consistent. If $p \sim q$, then $p \approx q$; but $p \approx q$ does not imply $p \sim q$ in general (e.g. in O_n, $n \geq 3$). But if A is stably finite, then $p \approx q$ if and only if $p \sim q$ ().

III.7.1.11. If an element x of a C*-algebra A is well-supported in the sense of (), then it is Cuntz equivalent to a projection; in fact, x is Cuntz equivalent to its left and right support projections. In particular, if x is a partial isometry, then x is Cuntz equivalent to its left and right support projections x^*x and xx^*. Conversely:

III.7.1.12. Proposition. Let A be a stably finite C*-algebra. Then an element $x \in A$ is Cuntz equivalent to a projection if and only if it is well-supported.

In contrast, if A is purely infinite, then any two nonzero elements of A are Cuntz equivalent (every nonzero hereditary C*-subalgebra of A contains a projection equivalent to 1_A).

Passage to Hereditary Subalgebras

III.7.1.13. If B is a C*-subalgebra of a C*-algebra A, and $a, b \in B$, then $b \precsim_B a$ implies $b \precsim_A a$, but not conversely in general (consider the case of projections). However, if B is a hereditary C*-subalgebra of A, then $\precsim_B$ and $\precsim_A$ coincide on B:

III.7.1.14. Proposition. Let A be a (local) C*-algebra and B a hereditary C*-subalgebra. If $a, b \in B$ and $b \precsim_A a$, then $b \precsim_B a$. Thus $\precsim_A$ and $\precsim_B$ coincide on B, and hence $\approx_A$ and $\approx_B$ also coincide on B.

Proof: Let $\epsilon > 0$. If $b \precsim_A a$, there are $x, y \in A$ with $\|xay - b\| < \frac{\epsilon}{3}$. Let (h_λ) be an approximate unit for B, with $\|h_\lambda\| \leq 1$ for all λ. Then $h_\lambda a h_\lambda \to a$ and $h_\lambda b h_\lambda \to b$ (). We also have

$$\|h_\lambda xay h_\lambda - h_\lambda b h_\lambda\| = \|h_\lambda(xay - b)h_\lambda\|\| \leq \|xay - b\| < \frac{\epsilon}{3}$$

for all λ and

$$\|h_\lambda x(h_\lambda a h_\lambda)y h_\lambda - h_\lambda xay h_\lambda\| = \|h_\lambda(xh_\lambda a h_\lambda y - xay)h_\lambda\| \leq \|x(h_\lambda a h_\lambda)y - xay\| < \frac{\epsilon}{3}$$

$$\|h_\lambda b h_\lambda - b\| < \frac{\epsilon}{3}$$

for sufficiently large λ, so for large λ,

$$\|h_\lambda(xh_\lambda a h_\lambda y)h_\lambda - b\| \leq \|h_\lambda(xh_\lambda a h_\lambda y)h_\lambda - h_\lambda xay h_\lambda\| + \|h_\lambda xay h_\lambda - h_\lambda b h_\lambda\| + \|h_\lambda b h_\lambda - b\| < \epsilon \ .$$

Since $h_\lambda x h_\lambda$ and $h_\lambda y h_\lambda$ are in B, and $\epsilon > 0$ is arbitrary, $b \precsim_B a$.

The Semigroup Operation

III.7.1.15. We can define a semigroup operation on equivalence classes. If A is a (local) C*-algebra, let $W(A)$ be the set of Cuntz equivalence classes of elements (positive elements) of $M_\infty(A)$. If $a \in M_\infty(A)$, denote by $\langle a\rangle$ the Cuntz equivalence class of a. (There is no collapsing going from A to $M_\infty(A)$ by III.7.1.14..) Note that if $a \in A_+$ (or a matrix algebra over A), then

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$$

via the elements

$$\begin{bmatrix} 0 & h_\lambda \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ h_\lambda & 0 \end{bmatrix}$$

where (h_λ) is an approximate unit for A (or the hereditary C*-subalgebra generated by a). Thus up to Cuntz equivalence we can move elements up or down the diagonal.

III.7.1.16. Now if $a, b \in A$, or a matrix algebra over A, define $\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle := \langle diag(a, b)\rangle$. This is easily seen to be well defined and to give an abelian semigroup structure on $W(A)$ with identity $\langle 0\rangle = \{0\}$. $W(A)$ is a *strict semigroup*: if $x + y = 0$, then $x = y = 0$. If $a, b \in A$ (or a matrix algebra), then

$$\begin{bmatrix} h_\lambda & h_\lambda \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} h_\lambda & 0 \\ h_\lambda & 0 \end{bmatrix} \to \begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix}$$

where (h_λ) is an approximate unit for A, so $a + b \precsim a \oplus b$. If $a \perp b$, then

$$\begin{bmatrix} h_n & 0 \\ k_n & 0 \end{bmatrix} \begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h_n & k_n \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where (h_n) and (k_n) are approximate units for the hereditary C*-subalgebras generated by a and b respectively, so $a \oplus b \precsim a + b$, and thus $a + b \approx a \oplus b$.

III.7.1.17. The semigroup $W(A)$ will be called the *basic* (or classical) *Cuntz semigroup* of A. There is inconsistency in the literature about what should be called the "Cuntz semigroup" of A, but it is now usually taken to be the often larger semigroup $W(A \otimes \mathbb{K})$ which has nicer properties. See III.7.5. for details. ✋

Ordering

III.7.1.18. There is a natural partial ordering on $W(A)$. If a, b are (positive) elements of $M_\infty(A)$, set $\langle b\rangle \le \langle a\rangle$ if $b \precsim a$. (Note that if $a_1 \approx a_2$ and $b_1 \approx b_2$, then $b_1 \precsim a_1$ if and only if $b_2 \precsim a_2$, so this is well-behaved.) If a_1, a_2, b_1, b_2 are (positive) elements of A (or a matrix algebra) and $\langle b_j\rangle \le \langle a_j\rangle$ $(j = 1, 2)$, then $\langle b_1\rangle + \langle b_2\rangle \le \langle a_1\rangle + \langle a_2\rangle$, so this ordering is compatible with the semigroup operation. And the argument of III.7.1.16. shows that $\langle a + b\rangle \le \langle a\rangle + \langle b\rangle$ for any a and b; $\langle a + b\rangle = \langle a\rangle + \langle b\rangle$ if $a \perp b$. "Ordering" on $W(A)$ will always mean this ordering unless otherwise specified.

III.7.1.19. The semigroup $W(A)$, like any abelian semigroup, has another (pre)ordering, the algebraic (pre)ordering (). This preordering, denoted $\le_a$, is in fact a partial ordering, since $\langle b\rangle \le_a \langle a\rangle$ implies $\langle b\rangle \le \langle a\rangle$. But the partial orderings $\le$ and $\le_a$ are different if A is unital, infinite-dimensional, and stably finite (III.7.3.4.). The algebraic ordering, as always, is also compatible with the semigroup operation.

Order Unit

III.7.1.20. If A is unital, there is an obvious order unit in $W(A)$, $\langle 1_A \rangle$. There is also an order unit whenever A is just σ-unital and, in particular, if A is separable: $\langle h \rangle$, where h is a strictly positive element of A. Any two strictly positive elements of A generate A as a hereditary C*-subalgebra, so are Cuntz equivalent by III.7.1.3.(iii), and if h is strictly positive then $a \precsim h$ for all $a \in A$. $h \oplus h \oplus \cdots \oplus h$ (n times) is strictly positive in $M_n(A)$ for any n, so $\langle h \rangle$ is an order unit in $W(A)$.

Functoriality

III.7.1.21. If A and B are C*-algebras, and $\phi : A \to B$ is a homomorphism (*-homomorphism), then since ϕ is continuous it preserves Cuntz comparison and equivalence, i.e. if $x, y \in A$ and $x \precsim_A y$, then $\phi(x) \precsim_B \phi(y)$. So there is a well-defined map $\phi_* : W(A) \to W(B)$ which preserves the natural partial ordering. It is obviously also a semigroup homomorphism (and therefore also preserves the algebraic ordering). Thus the Cuntz semigroup construction W is (covariantly) functorial.

III.7.1.22. The homomorphism ϕ_* is neither injective nor surjective in general. If ϕ is surjective, then ϕ_* is surjective. If ϕ is injective, then ϕ_* is frequently not injective; but if ϕ is the inclusion of a hereditary C*-subalgebra (e.g. an ideal), then ϕ_* is injective by III.7.1.14..

III.7.2. The Basic Cuntz Semigroup of a Commutative C*-algebra

III.7.2.1. To get a feel for Cuntz equivalence, consider the unital commutative case, $A = C(X)$ for a compact Hausdorff space X. If $f \in C(X)$, define the *cozero set* $coz(f)$ to be $\{x \in X : f(x) \neq 0\}$ (this set is sometimes called the *support* of f, but this term also sometimes means the closure of $coz(f)$ so is ambiguous). Then $f \precsim g$ in $C(X)$ if and only if $coz(f) \subseteq coz(g)$ (easy exercise using Urysohn's Lemma), so $f \approx g$ in $C(X)$ if and only if $coz(f) = coz(g)$. (Here we do not need to restrict to positive elements, which are just nonnegative real-valued continuous functions.) Thus the Cuntz equivalence classes in $C(X)$ are parametrized by the open F_σ's in X (just open sets if X is metrizable), and the ordering is by inclusion.

III.7.2.2. Things are more complicated in a matrix algebra over $C(X)$, say $M_n(C(X)) \cong C(X, \mathbb{M}_n)$. If $f, g \in C(X, \mathbb{M}_n)_+$, then a necessary condition for $f \precsim g$ is that the rank of $f(p)$ is less than or equal to the rank of $g(p)$ for every $p \in X$, so a necessary condition for $f \approx g$ is that f and g have the same rank at every point. The rank is a lower semicontinuous function from X to $\{0, 1, \ldots, n\}$. If X is metrizable, every such lower semicontinuous function is the rank function of an element of $M_n(C(X))$. Thus there is a surjective order-preserving function ρ from $W(C(X))$ to the set of bounded lower semicontinuous functions from X to $\mathbb{N} \cup \{0\}$, which is a homomorphism with respect to pointwise addition of functions.

III.7.2.3. From this we see that the ordering on $W(C(X))$ is not the algebraic ordering if X is infinite. A necessary condition that $\langle f \rangle \leq_a \langle g \rangle$ is that $\rho(\langle g \rangle) - \rho(\langle f \rangle)$ is nonnegative and lower semicontinuous, not always the case unless every open set in X is clopen, i.e. X is discrete. (Note that projections give continuous, i.e. locally constant, functions.) If $f, g \in C(X)$, then $\langle f \rangle \leq_a \langle g \rangle$ if and only if $coz(f)$ is a relatively closed subset of $coz(g)$. The situation is more complicated in matrix algebras, especially if ρ is not injective (note that ρ is always injective on classes of elements of $C(X)$).

III.7.2.4. If X is a well-behaved space of low dimension, e.g. $[0,1]$, it can be shown that ρ is injective (i.e. the necessary condition on ranks is also sufficient), hence a semigroup isomorphism and order-isomorphism. But if X is large enough to have a compact subspace supporting a nontrivial complex vector bundle (e.g. if X is a finite CW complex of dimension ≥ 3, which contains a 2-sphere), then ρ is not injective, even if X is topologically nice. For example, suppose $X = B^3$ (which is contractible and homeomorphic to $[0,1]^3$). Coordinatize X by spherical coordinates (r, ϕ, θ), with (ϕ, θ) parametrizing the unit sphere S^2. There is a nontrivial projection p in $M_2(C(S^2))$ of global rank 1 called the Bott projection (); write $p_{\phi,\theta}$ for the rank 1 projection in $\mathbb{M}_2$ at $(\phi, \theta) \in S^2$. Define $f : B^3 \to \mathbb{M}_2$ by

$$f(r, \phi, \theta) = rp_{\phi,\theta}$$

so $f \in M_2(C(B^3))_+$. We have that $\rho(f)$ is the function which is 0 at the origin and 1 elsewhere. However, f is not Cuntz equivalent to any element of $C(B^3)$.

We can modify this construction by letting g be any continuous function from $[0,1]$ to $[0,1]$ with $g(0) = 0$ and $g(1) = 1$, and setting $f_g(r, \phi, \theta) = g(r)p_{\phi,\theta}$. The zero-set of g can be any closed subset of $[0,1)$.We can even vary the vector bundle p on different intervals in $coz(g)$. No two of these elements are Cuntz equivalent. We can do a similar construction starting with any subset of B^3 homeomorphic to S^2 or, indeed, with any concentric sequence of homeomorphs of S^2, or with embedded tori, etc. Thus $W(C(B^3))$ is bewilderingly complicated. $W(C(X))$ is at least as complicated for any X containing a copy of B^3, since $C(B^3)$ is then a quotient of $C(X)$ so there is a surjective homomorphism from $W(C(X))$ to $W(C(S^3))$.

Even if $X = S^2$, where the Cuntz semigroup is well-behaved and computable, the map ρ is not injective. And the pathology is not just among well-supported elements: if f is a nonzero, noninvertible element of $C(S^2)$, then $[1] + [f]$ and $[p] + [f]$ are distinct elements of $W(C(S^2))$ not corresponding to well-supported elements (i.e. in the subsemigroup $W_0(C(S^2))$ defined below) on which ρ coincides.

If $dim(X)$ is even larger, $W(C(X))$ is even worse. For example, if X contains a 5-sphere (or 5-torus, etc.), lack of cancellation in $V(S^5)$ causes lack of cancellation in $W(C(X))$. Perforation already appears if X contains a 4-sphere or, more generally, a compact subspace with nontrivial H^4. Thus $W(C(X))$ is only reasonably behaved and can only be explicitly described for nice X of small dimension (zero- and one-dimensional, and some two-dimensional spaces; cf. [Rob13]). See [Tom08, 5.1] for further comments.

III.7.3. Two Subsemigroups of $W(A)$

III.7.3.1. Let A be a C*-algebra. There is an obvious homomorphism from $V(A)$ (whose elements are Murray-von Neumann equivalence classes of projections in $M_\infty(A)$) to $W(A)$, which is injective if A is stably finite. By slight abuse of notation, we will call the image subgroup $V(A)$ also (we are primarily interested in the stably finite case anyway, where this is not ambiguous).

III.7.3.2. We let $W_0(A) = W(A) \setminus V(A)$. If A is stably finite, $W_0(A)$ is exactly the set of equivalence classes of elements of $M_\infty(A)$ which are not well supported. Some references call this set $W(A)_+$, but this is not very good notation (). We have $W_0(A) = \emptyset$ if A is finite-dimensional, but every infinite-dimensional C*-algebra contains a positive element with infinite spectrum (), hence a positive element which is not well supported, so $W_0(A)$ is nonempty if A is infinite-dimensional and stably finite.

III.7.3.3. If $a, b \in M_\infty(A)_+$, then $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$, so if either a or b is not well-supported, neither is $a \oplus b$. Thus, if A is stably finite, $W_0(A)$ is a subsemigroup of $V(A)$. [This can fail in general: if A is a simple unital C*-algebra which is finite but not stably finite, cf. (), and a is an element of A which is not well-supported, then $\langle a \rangle \in W_0(A)$ but $n\langle a \rangle = n\langle 1_A \rangle$ for some sufficiently large n.]

III.7.3.4. So if A is stably finite, $W(A)$ is the disjoint union of the subsemigroups $V(A)$ and $W_0(A)$. The subsemigroup $W_0(A)$ absorbs $V(A)$: if $x \in V(A)$, $y \in W_0(A)$, then $x + y \in W_0(A)$. If A is unital, $V(A)$ is cofinal in $W(A)$ for the natural ordering (if A is stably projectionless, $V(A) = \{\langle 0 \rangle\}$); if A is infinite-dimensional and stably finite, then $W_0(A)$ is also cofinal in $W(A)$ (for the natural ordering). If A is infinite-dimensional and stably finite, then $V(A)$ is not cofinal in $W(A)$ for the algebraic ordering; in fact, we never have $y \leq_a x$ for $x \in V(A)$, $y \in W_0(A)$. In particular, the natural order and the algebraic order differ if A is unital, infinite-dimensional, and stably finite. If A is stably finite, the restriction of the (natural) ordering on $W(A)$ to $V(A)$ is the usual ordering on $V(A)$, which is the algebraic ordering. However, the restriction to $W_0(A)$ is not the algebraic ordering in general (III.7.4.2.).

III.7.3.5. If $A = C(X)$ is unital and commutative, then $V(C(X))$ is the set of $x \in W(C(X))$ for which $\rho(x)$ (III.7.2.2.) is continuous (i.e. locally constant), and $W_0(C(X))$ the set of $y \in W(C(X))$ for which $\rho(y)$ is discontinuous.

III.7.4. The Basic Cuntz Semigroup of a Simple AF Algebra

III.7.4.1. Example. Let A be a UHF algebra with $K_0(A)$ identified with a subgroup G of $\mathbb{Q}$ with $[1_A] = 1$ as usual. Then $V(A) \cong G_+ = G \cap [0, +\infty)$. Thus for each $r \in G_+$ we have an $x_r \in V(A) \subseteq W(A)$ which is the equivalence class of a projection of trace r in a matrix algebra over A. For each $r \in (0, +\infty)$ we construct an element y_r of $W_0(A)$ as follows. If $n \geq r$, there is a sequence of mutually orthogonal nonzero projections (p_k) in $M_n(A)$ with

$$\sum_{k=1}^{\infty} \tau(p_k) = r \ .$$

Set

$$a_r = \sum_{k=1}^{\infty} 2^{-k} p_k \in M_n(A)$$

and let $y_r = \langle a_r \rangle$.

It can be shown that every non-well-supported element of $M_\infty(A)$ is Cuntz equivalent to a_r for some unique r: in fact, if b is a non-well-supported positive element of A of norm one, then $b \approx a_r$, where

$$r = \sup_{n \in \mathbb{N}} \tau(b^{1/n})$$

(the fact that A has real rank zero, hence (HP), makes this easy) and thus $W_0(A) = \{y_r : r \in (0, +\infty)\}$. Note that if b is a projection (or well-supported positive element of norm one), then $\langle b \rangle = x_r$, where r is given by the same formula; in fact, this formula defines the unique dimension function () on A.

It is easy to see that $y_r + y_s = y_{r+s}$ for any $r, s \in (0, +\infty)$, and thus $W_0(A)$ is isomorphic to the additive semigroup $(0, +\infty)$. If $r \in G_+$, $s \in (0, \infty)$, then $x_r + y_s = y_{r+s}$. We have $x_r \leq x_s$ if and only if $r \leq s$ in G_+, and $y_r \leq y_s$ if and only if $r \leq s$ in $(0, +\infty)$. If $r \in G_+$, $s \in (0, +\infty)$, then $y_s \leq x_r$ if and only if $s \leq r$ and $x_r \leq y_s$ if and only if $r < s$. Thus $W(A)$ is isomorphic to $[0, +\infty)$ with the positive points of G "doubled," with the ordering a "lexicographic" order. (The algebraic ordering is different: we never have $y_s \leq_a x_r$ for any r,s.) Note that this semigroup does not have cancellation: we have $x_r + y_s = y_r + y_s$ for any $0 < r \in G_+$, $s \in (0, +\infty)$, but $x_r \neq y_r$.

III.7.4.2. Example. Let A be the simple unital AF algebra with dimension group $\mathbb{Q}^2$ with strict ordering and order unit $(1,1)$. As in III.7.4.1., every element of $W_0(A)$ is of the form $\langle a\rangle$, where

$$a=\sum_{n=1}^{\infty}2^{-n}p_n$$

for a sequence (p_n) of mutually orthogonal projections in $M_k(A)$ for some k. So $W_0(A)$ can be identified with (all) points in the open first quadrant of $\mathbb{R}^2$.

Let (p_n) and (q_n) $(n>2)$ be increasing sequences of projections in A with

$$[p_n]=\left(1-\frac{1}{n},\frac{1}{2}-\frac{1}{n}\right)$$

$$[q_n]=\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}-\frac{1}{2n}\right)$$

and $q_n\leq p_n$ for all n. Set $a=\sum_{n=3}^{\infty}2^{-n}p_n$ and $b=\sum_{n=3}^{\infty}2^{-n}q_n$. Then $b\leq a$ in A_+, so $\langle b\rangle\leq\langle a\rangle$ in $W_0(A)$. But there is no c with $\langle b\rangle+\langle c\rangle=\langle c\rangle$, so $\langle b\rangle\not\leq_a\langle a\rangle$ in $W_0(A)$. In fact in this case $W_0(A)$ is isomorphic to the open first quadrant in $\mathbb{R}^2$ with the *ordinary* ordering, not the algebraic ordering which is the strict ordering.

III.7.4.3. We now examine the case where A is a general infinite-dimensional simple unital AF algebra. The analysis works almost identically for any nonelementary separable simple C*-algebra of real rank zero and stable rank 1 once some technicalities are disposed of (). A similar but more complicated analysis works also in the nonsimple (real rank 0, stable rank 1) case.

Let A be a simple unital AF algebra. Recall that there is a homomorphism ρ from $V(A)$ to the additive semigroup $\{0\}\cup\mathrm{Aff}(\mathcal{T}(A))_{++}$ with $\rho([1_A])$ the constant function 1, where $\mathrm{Aff}(\mathcal{T}(A))_{++}$ is the set of continuous affine functions from $\mathcal{T}(A)$ to $(0,\infty)$. Let $\mathrm{Laff}(\mathcal{T}(A))_{++}$ be the additive semigroup of bounded lower semicontinuous functions from $\mathcal{T}(A)$ to $(0,\infty)$. We will define a semigroup order-isomorphism, also called ρ, from $W_0(A)$ to $\mathrm{Laff}(\mathcal{T}(A))_{++}$. This defines a homomorphism ρ from $W(A)=V(A)\cup W_0(A)$ onto $\{0\}\cup\mathrm{Laff}(A)_{++}$ which determines the ordering on $W(A)$.

III.7.4.4. Lemma. Let Δ be a metrizable Choquet simplex, and $f\in\mathrm{Laff}(\Delta)_{++}$. Then there is a strictly increasing sequence (f_n) in $\mathrm{Aff}(\Delta)_{++}$ (i.e. $f_n\ll f_{n+1}$ for all n) with $f=\sup f_n$.

Proof: Note that every lower semicontinuous function from Δ to $(0,\infty)$ is bounded below by compactness of Δ. If Δ is a Bauer simplex, the result is obvious from the well-known fact that a lower semicontinuous function on a compact metrizable space is a supremum of an increasing sequence of continuous functions, since $\mathrm{Aff}(\Delta)\cong C(\partial\Delta)$. In the general case, the sequence (f_n) can be made strictly increasing by subtracting small constant functions from the terms.

III.7.4.5. Now let $f \in \mathrm{Laff}(\mathcal{T}(A))_{++}$, with $f(\tau) \leq k$ for all $\tau \in \mathcal{T}(A)$. Then, since $\rho(V(A))$ is uniformly dense in $\mathrm{Aff}(\mathcal{T}(A))_{++}$ and ρ determines strict comparison in $V(A)$, there is a strictly increasing sequence (p_n) of projections in $M_k(A)$ such that $f = \sup \rho([p_n])$. Set

$$a = \sum_{n=1}^{\infty} 2^{-n} p_n \in M_k(A)_+ \ .$$

If (q_n) is another such sequence, for each n we have $\rho([q_n]) \ll \rho([p_m])$ for some m by Dini's Theorem ([?]), and hence $q_n \precsim p_m$, and conversely; thus, if $b = \sum_{n=1}^{\infty} 2^{-n} q_n$, then $b \approx a$; thus we obtain a well-defined equivalence class $y_f \in W_0(A)$. By the same argument, if $f, g \in \mathrm{Laff}(\mathcal{T}(A))_{++}$, $f \leq g$, we have $y_f \leq y_g$ in $W(A)$. Thus the map $\rho : y_f \mapsto f$ is a bijective order-preserving function.

III.7.4.6. If $a \in M_\infty(A)_+$, then the hereditary C*-subalgebra $\overline{aAa}$ has an increasing approximate unit (p_n) of projections since A has (HP). If a is not well supported, (p_n) can be taken to be strictly increasing, and it is easily verified that

$$a \approx \sum_{n=1}^{\infty} 2^{-n} p_n$$

so that $\langle a \rangle = y_f$, where $f = \sup \rho([p_n])$. Thus the map ρ is defined on all of $W_0(A)$, and is injective by the above argument. It is also easily seen to be a semigroup homomorphism.

III.7.4.7. To finish the description of $W(A)$, the map ρ, which is a semigroup homomorphism, determines the order on both $V(A)$ and $W(A)$. For the interaction, if $x \in V(A)$ and $y \in W_0(A)$, then $y \leq x$ if and only if $\rho(y) \leq \rho(x)$, and $x \leq y$ if and only if $\rho(x) \ll \rho(y)$. The algebraic ordering is quite different: no element of $W_0(A)$ is smaller than any element of $V(A)$, and unless A has unique trace the orderings are even different on $W_0(A)$, since if $y, z \in W_0(A)$, then $y \leq z$ if and only if $\rho(y) \leq \rho(z)$ but $y \leq_a z$ if and only if $\rho(y) = \rho(z)$ or $\rho(y) \ll \rho(z)$ and $\rho(z) - \rho(y)$ is lower semicontinuous.

Note that the map ρ is injective on $W_0(A)$ but not necessarily injective on $V(A)$. And it is never injective on $W(A)$ (for A nonelementary), since if $f \in \rho(V(A))$, there are $x \in V(A)$ and $y \in W_0(A)$ with $\rho(x) = \rho(y)$.

III.7.4.8. So, in the simple AF case (and, more generally in the simple real rank 0, stable rank 1 case), $W(A)$ can in a certain sense be regarded as a "monotone completion" of $V(A)$. This will be somewhat more accurate for the full Cuntz semigroup $\mathrm{Cu}(A)$ ().

III.7.5. The Full Cuntz Semigroup

III.7.5.1. We evidently have $W(M_n(A)) \cong W(A)$ for any A and n. More generally, if A and B are unital C*-algebras which are stably isomorphic (Morita equivalent), then $W(A) \cong W(B)$. However, if A is a C*-algebra, then $W(A \otimes \mathbb{K})$ is usually not isomorphic to $W(A)$; it is $W(A)$ with some "infinite" elements added in general. Thus the Cuntz semigroup is more sensitive to stabilization than K-theory (and does not behave as nicely for inductive limits). The semigroup $\mathrm{Cu}(A) = W(A \otimes \mathbb{K})$ is a "completed" version of $W(A)$, which does behave well under stabilization and even inductive limits. $\mathrm{Cu}(A)$ is called the *(full) Cuntz semigroup* of A.

III.7.5.2. Example. For a simple example, $W(\mathbb{C}) \cong \mathbb{N} \cup \{0\}$: the rank map is an isomorphism. The same is true for $W(\mathbb{K})$, except that $\mathbb{K}$ has elements of infinite rank, so $W(\mathbb{K}) \cong \bar{\mathbb{N}} := \{0\} \cup \mathbb{N} \cup \{\infty\}$ with $n + \infty = \infty$ for all n.

More generally, if X is a suitably nice compact metrizable space (e.g. a finite CW complex of dimension ≤ 1), then $W(C(X))$ is isomorphic to the semigroup of bounded lower semicontinuous functions from X to $\mathbb{N} \cup \{0\}$, and $\mathrm{Cu}(C(X))$ is isomorphic to the semigroup of lower semicontinuous functions from X to $\bar{\mathbb{N}}$.

III.7.5.3. The inclusion of A into $A \otimes \mathbb{K}$ as a hereditary C*-subalgebra (corner) induces a map from $W(A)$ into $\mathrm{Cu}(A)$ which is injective by III.7.1.14.. This inclusion can be surjective, e.g. if A is stable or purely infinite, but is "usually" not surjective.

Subsemigroups

It is convenient to slightly change the definition of the semigroup $W_0(A)$ (it is not a change in many cases).

III.7.5.4. If A is a stably finite C*-algebra, the Cuntz semigroup $\mathrm{Cu}(A)$ has three important subsemigroups:

The image of $V(A)$, the classes of projections (or well-supported elements) in $A \otimes \mathbb{K}$. Note that $V(A) = V(A \otimes \mathbb{K})$. We call this subsemigroup $V(A)$ also.

The set $W_0(A)$, the classes of (positive) elements of $Ped(A \otimes \mathbb{K})$ () which are not well-supported.

The set $\mathrm{Cu}_\infty(A)$, the remaining elements of $\mathrm{Cu}(A)$, i.e. the classes of (positive) elements of $A \otimes \mathbb{K}$ which are not in $Ped(A \otimes \mathbb{K})$; cf. ().

These subsemigroups are pairwise disjoint and their union is $\mathrm{Cu}(A)$.

We have that $V(A)$ and $W_0(A)$ are contained in the subsemigroup $W(A)$ of $\mathrm{Cu}(A)$. In general there can be elements of $W(A)$ which are in $\mathrm{Cu}_\infty(A)$; but if A is unital or algebraically simple, then $W(A)$ is the disjoint union of $V(A)$ and $W_0(A)$, which agrees with the previously defined $W_0(A)$.

III.7.5.5. Each of these subsemigroups absorbs the previous ones:

$V(A) + W_0(A) \subseteq W_0(A)$.

$W_0(A) + \mathrm{Cu}_\infty(A) \subseteq \mathrm{Cu}_\infty(A)$.

$V(A) + \mathrm{Cu}_\infty(A) \subseteq \mathrm{Cu}_\infty(A)$.

The subsemigroup $V(A) \cup W_0(A)$ is hereditary (an order ideal): if $y \in V(A) \cup W_0(A)$, $x \in \mathrm{Cu}(A)$ with $x \leq y$, then $x \in V(A) \cup W_0(A)$.

III.7.5.6. EXAMPLES. (i) Let X be a finite CW complex of dimension ≤ 1. Then $\mathrm{Cu}(C(X))$ is isomorphic to the semigroup of lower semicontinuous functions from X to $\bar{\mathbb{N}}$ (). Under this identification, $V(C(X))$ is the set of continuous (locally constant) functions from X to $\mathbb{N}\cup\{0\}$, $W_0(C(X))$ the set of bounded discontinuous but lower semicontinuous functions from X to $\mathbb{N}\cup\{0\}$, and $\mathrm{Cu}_\infty(C(X))$ the set of lower semicontinuous functions which are either unbounded or take the value ∞ somewhere.

(ii) Let A be the simple unital AF algebra of III.7.4.2.. Then

$$\mathrm{Cu}(A) \cong \{(0,0)\} \cup \mathbb{Q}^2_{++} \cup (0,\infty] \times (0,\infty]$$

where $\mathbb{Q}^2_{++}$ is the set of points of $\mathbb{Q}^2$ with strictly positive coordinates. Under this identification, we have

$$V(A) = \{(0,0)\} \cup \mathbb{Q}^2_{++} \ .$$

$$W_0(A) = (0,\infty) \times (0,\infty) \ .$$

$$\mathrm{Cu}_\infty(A) = (0,\infty] \times \{\infty\} \cup \{\infty\} \times (0,\infty] \ .$$

There is a largest element (∞,∞) in $\mathrm{Cu}(A)$, which is in $\mathrm{Cu}_\infty(A)$.

III.7.5.7. Actually, there is a largest element of $\mathrm{Cu}(A)$ whenever A is σ-unital (e.g. separable): $\langle h\rangle$, where h is strictly positive in $A\otimes\mathbb{K}$. This element is also the largest element for the algebraic ordering, i.e. $\langle a\rangle + \langle h\rangle = \langle h\rangle$ for any $a \in A\otimes\mathbb{K}$.

Countable Sums in $\mathrm{Cu}(A)$

III.7.5.8. The (full) Cuntz semigroup allows arbitrary countable sums as well as finite sums. If (x_k) is a sequence in $\mathrm{Cu}(A)$, we can form $\sum_{k=1}^\infty x_k$ in $\mathrm{Cu}(A)$ as follows. For each k, let $x_k = \langle a_k\rangle$, where $a_k \in (A\otimes\mathbb{K})_+$ and $\|a_k\| \leq 2^{-k}$. Then $a_1 \oplus a_2 \oplus \cdots$ defines a positive element a of $(A\otimes\mathbb{K})\otimes\mathbb{K} \cong A\otimes\mathbb{K}$ (via the natural isomorphism of $(A\otimes\mathbb{K})\otimes\mathbb{K}$ with $A\otimes(\mathbb{K}\otimes\mathbb{K})$). Set $x = \langle a\rangle$. If (b_k) is another such sequence defining b, then for each n

$$b_1 \oplus \cdots \oplus b_n \approx a_1 \oplus \cdots \oplus a_n \precsim a$$

and $b_1 \oplus \cdots \oplus b_n \to b$, so $b \precsim a$ by III.7.1.9.; similarly $a \precsim b$, so $\langle a\rangle = \langle b\rangle$ and x is well defined. Similarly, if $a_1 \oplus \cdots \oplus a_n \precsim c$ for all n, then $a \precsim c$ since $a_1 \oplus \cdots \oplus a_n \to a$. Thus x is the least upper bound of the sequence $(\sum_{k=1}^n x_k)$, so we may legitimately write

$$x = \sum_{k=1}^\infty x_k \ .$$

If (b_k) is any sequence of pairwise orthogonal positive elements of $A\otimes\mathbb{K}$ with $x_k = \langle b_k\rangle$ for all k and $\|b_k\| \to 0$, so $b = \sum_{k=1}^\infty b_k$ exists in $A\otimes\mathbb{K}$, then $x = \langle b\rangle$. The infinite sum x is typically in $\mathrm{Cu}_\infty(A)$, but it can be in $W_0(A)$; this will happen if there is a $y \in W_0(A)$ with $x_1 + \cdots + n \leq y$ for all n, so $x \leq y$ The x can never be in $V(A)$ unless only finitely many x_k are nonzero.

The Group $K_0^*(A)$

III.7.5.9. We can form the Grothendieck group () of the semigroup $W(A)$, called $K_0^*(A)$ in [Cun78]. $K_0^*(A)$ has a natural ordering induced from $W(A)$ making it an ordered group (in the stably finite case); in fact, the positive cone is exactly the image of $W(A)$ in good cases (not always!). If A is unital, $(K_0^*(A), K_0^*(A)_+)$ has a natural order unit [1], and $(K_0^*(A), K_0^*(A)_+, [1])$ is a scaled ordered group (usually not countable, even if A is separable). For example, if A is any UHF algebra, then $K_0^*(A)$ is algebraically and order-isomorphic to the additive group $\mathbb{R}$. When forming $K_0^*(A)$, it is crucial to work only with $W(A)$ and not the completion $\mathrm{Cu}(A)$ with infinite elements (which would force the Grothendieck group to be degenerate). It was natural to work with $K_0^*(A)$ for the purposes of [Cun78], but much information is usually lost in going from $W(A)$ or $\mathrm{Cu}(A)$ to $K_0^*(A)$ (for example, $K_0(A)$ largely disappears). The functor K_0^* is also not continuous () and does not respect stabilization or general Morita equivalence.

III.7.6. The Cuntz Semigroup via Hilbert Modules

In [**?**], the Cuntz semigroup $\mathrm{Cu}(A)$ was recast in terms of Hilbert A-modules. Although this approach requires some additional background, it gives a very clean description of the Cuntz semigroup which makes some crucial properties much more transparent. For the definition and general theory of Hilbert modules, see [Bla06, II.7], [Bla98], [**?**], [**?**], [**?**].

In this section, all C*-algebras will be *separable*. We will use B as our generic C*-algebra so the notation is consistent with [Bla06], since we use right modules.

III.7.6.1. The motivation for this approach is that if R is a unital ring, then $V(R)$ can be naturally thought of as the set of isomorphism classes of finitely generated projective (right) R-modules, with direct sum as semigroup operation (cf. [Bla98,]). If B is a unital C*-algebra, then any finitely generated projective (right) B-module has an essentially unique structure as a Hilbert B-module. We generalize this by allowing countably generated Hilbert B-modules which are not necessarily projective, and also allow B to be nonunital.

III.7.6.2. Let B be a (separable) C*-algebra. Let $\mathrm{Hil}(B)$ be the set of isomorphism classes of countably generated (right) Hilbert B-modules (isomorphism means isometric module-isomorphism). $\mathrm{Hil}(B)$ becomes an abelian semigroup under the operation of direct sum, with the 0-module as unit. There is a preorder $\precsim$ on $\mathrm{Hil}(B)$, where $\mathcal{X} \precsim \mathcal{Y}$ if X is isomorphic to a submodule of $\mathcal{Y}$ (not necessarily complemented, i.e. the inclusion map of $\mathcal{X}$ into $\mathcal{Y}$ is not necessarily adjointable). Set $\mathcal{X} \approx \mathcal{Y}$ if $\mathcal{X} \precsim \mathcal{Y}$ and $\mathcal{Y} \precsim \mathcal{X}$, i.e. each is isomorphic to a submodule of the other. Let $\widetilde{\mathrm{Hil}(B)}$ be the set of $\approx$-equivalence classes; then $\precsim$ induces a partial order $\leq$ on $\widetilde{\mathrm{Hil}(B)}$, and direct sum drops to a semigroup operation on $\widetilde{\mathrm{Hil}(B)}$.

III.7.6.3. Theorem. ([**?**], []) The semigroup $\widetilde{\mathrm{Hil}(B)}$ is naturally isomorphic to $\mathrm{Cu}(B)$.

III.7.6.4. The correspondence goes like this.

As a closely related matter, we have the following version of ():

III.7.6.5. Proposition. Let B be a stably finite C*-algebra, and $\mathcal{X}$ and $\mathcal{Y}$ Hilbert B-modules. If $\mathcal{X} \approx \mathcal{Y}$, then $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic. Thus $\mathrm{Cu}(B) \cong \mathrm{Hil}(B)$.

III.7.6.6. The three subsemigroups (in the stably finite case) of $\mathrm{Cu}(B)$ can be nicely described in Hilbert module terms. We say a Hilbert B-module $\mathcal{X}$ is *compact* if $\mathcal{L}(\mathcal{X}) = \mathcal{K}(\mathcal{X})$. A compact Hilbert B-module is finitely generated; if B is unital, then a Hilbert B-module is compact if and only if it is finitely generated and projective. If B is stably finite, we have:

$V(B)$ is the set of classes of compact Hilbert B-modules.

$W_0(B)$ is the set of classes of noncompact finitely generated Hilbert B-modules.

$\mathrm{Cu}_\infty(B)$ is the set of classes of countably generated Hilbert B-modules which are not finitely generated.

There is a "universal" countably generated Hilbert B-module, $\mathcal{H}_B = B^\infty$, which absorbs all countably generated Hilbert B-modules as direct summands by the Stabilization Theorem [Bla98,]. This Hilbert B-module corresponds to the largest element of $\mathrm{Cu}(B)$ described in III.7.5.7..

III.7.7. Abstract Cuntz Semigroups

III.7.8. Augmented Cuntz Semigroups

Chapter IV

Construction of Examples

There are explicit constructions of examples of classifiable C*-algebras with all possible Elliott invariants. The stably finite case is much more delicate than the purely infinite case, which is treated in IV.4.; in the stably finite case, every value for the Elliott invariant which satisfies the properties of III.4.2.1., and in which the K_0 group is weakly unperforated (which will be automatic for classifiable algebras, cf. ()), occurs for a simple unital ASH algebra with base spaces of dimension ≤ 2. The construction also gives stably projectionless examples, and in fact every classifiable stably projectionless C*-algebra up to stable isomorphism (although it takes some effort to show this, cf. ()). The general construction theorem is stated in IV.3.5.1..

The general construction is described in [Ell96], and in more detail (and perhaps more accurately) in [**?**]. The general construction is quite delicate and requires over 100 pages of careful calculations and estimates, so will not be repeated here. This is, in fact, perhaps the part of the classification program most in need of simplification (maybe a crossed product or groupoid construction can be used to generate the examples more easily, although such an approach will probably not exhibit them as ASH algebras). We will satisfy ourselves with explicit constructions of some particularly important and natural examples, which will together illustrate the techniques needed for the general construction. The examples will cover both the unital and nonunital cases.

There are *ad hoc* constructions which have been used in various special cases (e.g. I.3.5.4.). The constructions described here follow a uniform pattern and will work for all (weakly unperforated) possibilities for the Elliott invariant, so in particular will work to get all examples (in retrospect!) of simple AF, AI, and A$\mathbb{T}$ algebras as well as the Jiang-Su algebra $\mathcal{Z}$, although they are not always the most elementary or efficient ones in these special cases.

IV.1. Examples with Trivial K-Theory

IV.1.1. Building Blocks

IV.1.2. The Basic Example

IV.1.2.1. We begin with a natural example: a (necessarily nonunital and even stably projectionless) simple ASH algebra A with $K_0(A) = K_1(A) = 0$ and with unique trace (there will actually be two such examples, one with finite trace and a stable one with unbounded trace, which are stably isomorphic. We will call the finite one $\mathcal{W}$ and the stable one Ω.) Once we have this example, we may more generally construct an example with trivial K-theory and arbitrary trace cone C with base Δ (an arbitrary metrizable Choquet simplex) by taking $B \otimes \Omega$, where B is any simple AF algebra with $\mathcal{T}^+(B) \cong C$ (cf. I.3.3.58.; we are of course not concerned with uniqueness at this point); a Bratteli diagram for B, combined with the system for Ω, gives an explicit inductive system for $B \otimes \Omega$ with the same type of building blocks as for Ω.

Even the construction of this example is fairly delicate. If we were not concerned with making the algebra have unique trace, the construction could be done more simply.

This example was first constructed in [Bla80], and then again later as an ASH algebra in [Raz02]–[Jac13]; the construction here is similar to the one in [Jac13]. The construction in [Bla80] (which is simpler but not as elementary and uses some nontrivial structure results) does not exhibit $\mathcal{W}$ as an ASH algebra, and it is not obvious, and has not been widely noted, that the two constructed algebras are isomorphic; in fact, the isomorphism is only easily seen using the classification theorem. (Actually, the author himself did not notice the isomorphism until recently; when [Bla80] was written, he did not know K-theory so he didn't compute it!)

IV.1.2.2. For completeness, we outline the construction of [Bla80], omitting some technicalities. Let B be the (unique) simple unital AF algebra whose dimension group is the additive group G of real algebraic numbers (the additive group of any countable subfield of $\mathbb{R}$ in which every positive element has a square root would do as well). B has unique trace τ. If $\alpha \in G \cap (0,1)$, and p is a projection in B with trace α, then $pBp \cong B$; let σ_α be an isomorphism from B onto pBp, regarded as an endomorphism of B (any two such p and σ_α are homotopic and approximately unitarily equivalent as endomorphisms of B).

Form the mapping torus

$$M_\alpha = \{f : [0,1] \to B : f(1) = \sigma_\alpha(f(0))\}$$

of σ_α. Then M_α is nonunital and projectionless: if q is a projection in M_α, then $q = (q(t))$, where $q(t)$ is a projection in B for each $t \in [0,1]$. Then $\tau(q(t))$ must be constant since the $q(t)$ are all equivalent (); but $\tau(q(1)) = \alpha\tau(q(0))$, so $q(t) = 0$ for all t.

For any such α there is a "twice-around" embedding of M_α into $M_{\sqrt{\alpha}}$ defined similarly to the embedding in ().

Fix an $\alpha_1 \in G \cap (0,1)$ and inductively define $\alpha_{n+1} = \sqrt{\alpha_n}$. We then have an inductive system

$$M_{\alpha_1} \longrightarrow M_{\alpha_2} \longrightarrow \cdots$$

where the connecting maps are the twice-around embeddings. Let A be the inductive limit.

IV.1.2.3. Proposition. A is simple, UCT, has nuclear dimension 1, with unique trace, and $K_0(A) = K_1(A) = 0$.

Proof: (Outline) The proof that A is simple with unique trace is very similar to the argument in (). Each M_{α_n} is an extension of an AF algebra by the suspension of an AF algebra, hence is UCT, so A is also UCT. The nuclear dimension of each M_{α_n} is ≤ 2, hence $dim_{nuc}(A) \leq 2$; by () it must be ≤ 1 (actually $dim_{nuc}(M_{\alpha_n}) = 1$, and $dim_{nuc}(A) \geq 1$ since A, being projectionless, is clearly not AF).

To prove that A has trivial K-theory, it suffices to show that M_α has trivial K-theory for any $\alpha \in G \cap (0,1)$.

Such M_α is an extension of B by SB, so the six-term exact sequence of K-theory gives

$$\begin{tikzcd}
K_0(SB) \arrow[r] & K_0(M_\alpha) \arrow[r] & K_0(B) \arrow[d, "\delta"] \\
K_1(B) \arrow[u] & K_1(M_\alpha) \arrow[l] & K_1(SB) \arrow[l]
\end{tikzcd}$$

We have $K_0(B) = K_1(SB) = G$ and $K_1(B) = K_0(SB) = 0$, so we obtain the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K_0(M_\alpha) \arrow[r] & G \arrow[d, "\delta"] \\
0 \arrow[u] & K_1(M_\alpha) \arrow[l] & G \arrow[l]
\end{tikzcd}$$

and the connecting map δ is multiplication by $1-\alpha$ (the argument is almost identical to the one in ()). Since G is the additive group of a field and $\alpha < 1$, this map is an isomorphism, and it follows that $K_0(M_\alpha) = K_1(M_\alpha) = 0$.

IV.1.3. Weakly Self-Absorbing Algebras

IV.1.3.1. This $\mathcal{W}$ has some properties similar to the properties of the Jiang-Su algebra $\mathcal{Z}$; in fact, it should play a similar role for classifiable algebras with trivial K-theory. $\mathcal{W}$ is not SSA (the definition does not make sense for nonunital algebras), but it does satisfy $\mathcal{W} \cong \mathcal{W} \otimes \mathcal{W}$ (the classification theorem implies this; maybe it can be proved directly), and it should be true that $\mathcal{W}$ has approximately inner flip (i.e. approximately multiplier inner flip).

IV.1.3.2. Definition. Let A be a C*-algebra. A is *weakly self-absorbing (WSA)* if

$A \cong A \otimes A$.

A has approximately inner flip (approximately multiplier inner flip in the nonunital case).

$A \not\cong \mathbb{C}$.

IV.1.3.3. A WSA C*-algebra is simple and nuclear, and either has unique trace or is purely infinite [ER78]. If A is WSA and in the UCT class, then $K_1(A) = 0$ and $K_0(A)$ is torsion-free of rank ≤ 1.

IV.1.3.4. Examples. (i) Any SSA C*-algebra (I.6.5.5.).

(ii) $\mathbb{K}$.

(iii) Any tensor product of WSA C*-algebras; in particular, the stabilization of any SSA C*-algebra.

IV.1.3.5. Question. Is any unital WSA C*-algebra SSA? Are $\mathcal{W}$ and Ω WSA? What are all the WSA C*-algebras?

IV.2. Examples With the K-Theory of $\mathbb{C}$

We now look at the possibilities of ASH algebras A with $K_*(A) = K_*(\mathbb{C})$, i.e. $K_0(A) = \mathbb{Z}$ and $K_1(A) = 0$. We will use slightly more complicated building blocks than the ones in (). The examples will be both unital and stably projectionless.

IV.2.1. A Generalization of $\mathcal{Z}$

Next we construct simple unital ASH algebras with the same K-theory as $\mathbb{C}$ but with an arbitrary trace simplex. This construction with a one-point trace simplex gives an alternate construction of the Jiang-Su algebra $\mathcal{Z}$.

IV.2.2. A Stably Projectionless Version of $\mathbb{C}$ with Unique Trace

Call it $\mathcal{K}$. For which B is $\mathcal{K} \otimes B \cong B$?

IV.2.3. Stably Projectionless Versions of $\mathbb{C}$ with Nonunique Trace

IV.2.3.1. A simple C*-algebra A with $K_0(A) = \mathbb{Z}$ and unique trace can have essentially only two (unperforated) pairings of $K_0(A)$ with the trace: trivial and nontrivial. If the pairing is trivial, as in (), A is stably projectionless (and if classifiable is isomorphic to one of the two algebras of ()). And if the pairing is nontrivial, there is a generator u of $K_0(A)$ which is positive, hence corresponds to a projection in $M_\infty(A)$. If A is classifiable, it is then isomorphic to either a matrix algebra over $\mathcal{Z}$ if unital or a hereditary subalgebra of $\mathcal{Z} \otimes \mathbb{K}$ if nonunital; there is (up to isomorphism) a sequence of algebras of the first type parametrized by $\mathbb{N}$, and a continuum of the second type parametrized by $(0, +\infty]$ similarly to I.3.3.62.(i).

IV.2.3.2. But there is more leeway if there is more than one trace, even if there are two extremal traces. Suppose $\mathcal{T}(A) = [-1, 1]$ (i.e. there are two extremal traces, both of which we assume to be finite for now). If u is a generator for $K_0(A)$, then up to rescaling and symmetry (i.e. in the classifiable case, up to stable isomorphism) there are, replacing u by $-u$ if necessary, four possibilities for the values $\rho_A(u)$ takes on the extreme traces:

(i) $\rho_A(u) = (1, 1)$.

(ii) $\rho_A(u) = (0, 0)$.

(iii) $\rho_A(u) = (1, 0)$.

(iv) $\rho_A(u) = (1, -1)$.

An algebra in case (i) is stably isomorphic to a unital algebra (if classifiable, stably isomorphic to the algebra of ()), while ones in (ii)–(iv) are stably projectionless. There are three classifiable isomorphism classes in case (ii), all stably isomorphic to the algebra of () (depending on how many of the extremal traces are finite). In cases (iii) and (iv), distinct stable isomorphism classes are obtained; within each case the classifiable algebras are stably isomorphic, but there are many isomorphism classes in each stable isomorphism class. We will construct an example in each case; all the others can be obtained from these by amplification and reduction.

IV.3. More Complicated K-Theory

IV.3.1. Examples With Torsion-Free K_0 and Trivial K_1

These examples will include all simple AI algebras (in particular, all simple AF algebras), as well as examples like the ones in I.3.7.14. and I.3.7.15., and versions with fewer or no projections.

The ones with trivial pairing of K_0 with the traces are easily obtained as in IV.1.2.1., tensoring the example of () with a simple AF algebra or AI algebra B with suitable K_0 and trace space (there is much flexibility here since the pairing of $K_0(B)$ with $\mathcal{T}(B)$ is irrelevant). But there are many more projectionless algebras in this category with nontrivial pairing which cannot be constructed this way. In any event, it is desirable to have a unified construction of all examples, even including simple AI algebras (in principle, the general construction should even include simple AF algebras, but in practice it is convenient to begin with the AF construction from the Effros-Handelman-Shen Theorem and build the general construction on that).

IV.3.2. Examples With Nontrivial K_1

IV.3.3. Dimension Groups With Torsion

As a preliminary to constructing examples with torsion in K_0, we need to obtain a version of the Effros-Handelman-Shen Theorem (I.3.3.47.) which includes groups with torsion.

IV.3.4. Obtaining Torsion in K_0

The most difficult and complicated stably finite examples to construct are ones with torsion in K_0. To get such ASH algebras, we must work over base spaces of dimension two (for AH examples, which we do not treat here, we must even use base spaces of dimension three).

IV.3.5. The General Stably Finite Case

The general construction theorem is:

IV.3.5.1. Theorem. Let G_0 and G_1 be countable abelian groups, Δ a metrizable Choquet simplex, and $\rho : G_0 \to \text{Aff}(\Delta)$ a homomorphism. Then there is a simple ASH algebra A with base spaces of dimension ≤ 2, with no unbounded traces, constructed by a uniform scheme from the data, with $K_*(A) \cong (G_0, G_1)$, and an isomorphism of $\mathcal{T}(A)$ with Δ intertwining ρ with the pairing of $K_0(A)$ with $\mathcal{T}(A)$. If there is $u \in G_0$ with $\rho(u)$ the constant function 1, A may be constructed by the scheme so that it is unital and $u = [1_A]$ under the correspondence.

Using the general construction theorem, we can painlessly obtain many examples including ones previously constructed laboriously.

IV.3.5.2. Examples. In all of the following examples, we will take K_1 to be 0.

(i) Embed $\mathbb{Q}^3$ into $\mathbb{R}^4 \cong \text{Aff}(\Delta_3)$ by

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, x_1 + x_3 - x_2)$$

and give $\mathbb{Q}^3$ the strict ordering from this map. Taking $(1,1,1,1)$ as order unit, we get a unital classifiable algebra A with four extremal traces, essentially example I.3.7.14. (precisely example I.3.7.14. tensored with the universal UHF algebra), for which the state space of $K_0(A)$ is a square.

More generally, we may take any countable subgroup G of $\mathbb{R}^4$ containing $(1,1,1,1)$, with the property that $x_1+x_3=x_2+x_4$ for any $(x_1,x_2,x_3,x_4)\in G$. For example, we can take any countable subgroups G_1,G_2,G_3 of $\mathbb{R}$ containing 1, and let G be

$$(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : x_1\in G_1, x_2\in G_2, x_3\in G_3, x_4=x_1+x_3-x_2\}\ .$$

The G_j do not have to be dense in $\mathbb{R}$, cf. IV.3.5.3.(iii).

(ii) Embed $\mathbb{Q}^2$ in $\mathbb{R}^3\cong \text{Aff}(\Delta_2)$ by

$$(x_1,x_2)\mapsto\left(x_2,\frac{x_1+x_2}{2},x_2\right)$$

and give $\mathbb{Q}^2$ the strict ordering from this map. Taking $(1,1)$ as order unit, we get a unital classifiable algebra A with three extremal traces, essentially example I.3.7.15. (precisely example I.3.7.15. tensored with the universal UHF algebra). The state space of $K_0(A)$ is an interval, but the map from $\mathcal{T}(A)$ to this state space does not send extreme points to extreme points.

Unital Projectionless Examples

Here are a few simple but interesting examples of unital projectionless classifiable C*-algebras.

IV.3.5.3. Examples. In all of the following examples, we will again take K_1 to be 0.

(i) Take $\mathbb{Z}^2$ with the strict ordering. This fits into the scheme of () by taking the identity map of $\mathbb{Z}^2$ into $\mathbb{R}^2\cong\text{Aff}(\Delta_1)$. Let A be the corresponding classifiable algebra with order unit $(1,1)$. Then A is projectionless, and has two extremal traces τ_1 and τ_2. $M_2(A)$ has exactly one equivalence class of nontrivial projections, $[diag(1_A,0)]$. But $M_3(A)$ has two "unexpected" equivalence classes of projections corresponding to $(2,1)$ and $(1,2)$ in addition to the two "expected" equivalence classes $[diag(1_A,0,0)]$ and $[diag(1_A,1_A,0)]$ corresponding to $(1,1)$ and $(2,2)$ respectively. If τ_1 and τ_2 are extended to (unnormalized) traces on $M_3(A)$ with $\tau_1(1_{M_3(A)})=\tau_2(1_{M_3(A)})=3$, and p is a projection in the class of $(1,2)$, then $\tau_1(p)=1$ and $\tau_2(p)=2$, and the reverse for a projection in the equivalence class $(2,1)$.

Let B be the corner of $M_3(A)$ corresponding to $(2,1)$. Then B is stably isomorphic to A and is also projectionless. $M_2(B)$ has two unexpected equivalence classes of nontrivial projections corresponding to $(1,1)$ and $(3,1)$ in addition to the expected class $[diag(1_B,0)]$ corresponding to $(2,1)$ (the identity of $M_2(B)$ of course corresponds to $(4,2)$). A is isomorphic to a corner in $M_2(B)$. B also has two extremal traces σ_1 and σ_2 (σ_1 is essentially $\tau_1/2$ and σ_2 is essentially τ_2). If σ_1 and σ_2 are extended to (unnormalized) traces on $M_2(B)$ taking the value 2 on the unit, then for p corresponding to $(1,1)$ we have $\sigma_1(p)=1/2$ and $\sigma_2(p)=1$, and for q corresponding to $(3,1)$ we have $\sigma_1(q)=3/2$ and $\sigma_2(q)=1$.

(ii) Take $\mathbb{Z}^2$ with the strict ordering from the first coordinate. This fits the scheme by taking the map from $\mathbb{Z}^2$ to $\mathbb{R}\cong\text{Aff}(\Delta_0)$ sending (n_1,n_2) to n_1. Let A be the corresponding classifiable algebra with order unit $(1,0)$ (A has the same scaled ordered K_0 as $C(S^2)$). Then A is projectionless, with unique trace τ. But $M_2(A)$ has infinitely many equivalence classes of projections, corresponding to $(1,n)$ for any $n\in\mathbb{Z}$. If τ is extended to $M_2(A)$ taking the value 2 on the unit, then $\tau(p) = 1$ for any nontrivial projection in $M_2(A)$. We have $pM_2(A)p \cong A$ for any nontrivial projection p in $M_2(A)$.

We may modify this example by instead taking Δ to be any metrizable Choquet simplex, and mapping $\mathbb{Z}^2$ into $\mathrm{Aff}(\Delta)$ by sending (n_1, n_2) to the constant function n_1. Again taking $(1, 0)$ as order unit, we get a classifiable algebra A_Δ with similar properties, in particular projectionless, with $\mathcal{T}(A_\Delta) \cong \Delta$. Although A_Δ has many traces, they all agree on projections in any matrix algebra.

(iii) Embed $\mathbb{Z}^3$ into $\mathbb{R}^4 \cong \mathrm{Aff}(\Delta_3)$ by

$$(n_1, n_2, n_3) \mapsto (n_1, n_2, n_3, n_1 + n_3 - n_2)$$

and give $\mathbb{Z}^3$ the strict ordering from this map. Taking $(1, 1, 1, 1)$ as order unit, we get a projectionless unital classifiable algebra A with four extremal traces. The state space of $K_0(A)$ is a square, similar to I.3.7.14. and IV.3.5.2.(i). We may also make a projectionless example similar to IV.3.5.2.(ii).

In all the above examples, we may take K_1 to be any specified countable abelian group instead of 0. We can also clearly make enormously more complicated examples along the same lines.

IV.4. The Purely Infinite Case

IV.4.1. Introduction

IV.4.1.1. For the purely infinite case, we simply need to show that if G_0 and G_1 are countable abelian groups, there is a UCT Kirchberg algebra A with $K_0(A) \cong G_0$ and $K_1(A) \cong G_1$, for then the position of the unit in K_0 can be made arbitrary in $K_0(A)$ by taking a suitable corner (cf. III.2.1.6.).

IV.4.1.2. Once the ASH construction has been done, there is a quick way to do the purely infinite case: given countable abelian groups G_0 and G_1, construct a simple ASH algebra A with $K_0(A) \cong G_0$ and $K_1(A) \cong G_1$, and take $A \otimes O_\infty$ (). We do not even need the full force of the ASH construction since the ordering on G_0 is irrelevant. If we stick to unital ASH algebras, we only get examples where G_0 has a subgroup isomorphic to $\mathbb{Z}$ (i.e. is not a torsion group), but we can get arbitrary G_0 and G_1 using stable ASH algebras (note that if A is a stable simple ASH algebra, then by [BC82] $A \otimes O_\infty$ contains an infinite projection, hence has a corner which is a UCT Kirchberg algebra with the same K-theory).

IV.4.2. A Quick Approach from Classification

IV.4.2.1. But there are easier ways to do it. We will give a simple construction which uses the Kirchberg-Phillips classification. The construction can be done in at least two lower-tech but somewhat more complicated ways which do not use the Kirchberg-Phillips classification; see [RS02].

We specifically use the following two facts from Kirchberg-Phillips:

IV.4.2.2. Theorem. Let A and B be UCT Kirchberg algebras. If $\phi_0 : K_0(A) \to K_0(B)$ and $\phi_1 : K_1(A) \to K_1(B)$ are homomorphisms with $\phi_0([1_A]) = [1_B]$, then there is a unital embedding $\phi : A \to B$ which induces ϕ_j on K_j $(j = 0, 1)$.

IV.4.2.3. Corollary. Let A be a UCT Kirchberg algebra. Then A embeds unitally in O_2.

The corollary also follows from (), or from the somewhat more elementary (), since there is an obvious unital embedding $a \mapsto a \otimes 1$ from A to $A \otimes O_2 \cong O_2$.

IV.4.2.4. We need as building blocks the following UCT Kirchberg algebras:

O_{n+1}: $K_0(O_{n+1}) = \mathbb{Z}_n$, $K_1(O_{n+1}) = 0$.

O_∞: $K_0(O_\infty) = \mathbb{Z}$, $K_1(O_\infty) = 0$.

O_{1^-} (I.5.1.41.): $K_0(O_{1^-}) = \mathbb{Z}$, $K_1(O_{1^-}) = \mathbb{Z}$ (we actually do not need O_{1^-}, but see IV.4.2.5.).

P_{n+1} (I.5.1.45.(i)): $K_0(P_{n+1}) = 0$, $K_1(P_{n+1}) = \mathbb{Z}_n$.

IV.4.2.5. We also need one more building block. Let A be a corner in O_{1^-} with $[1_A] = 0$ in $K_0(A) \cong \mathbb{Z}$. By IV.4.2.2., there is a unital embedding $\phi : A \to A$ inducing ϕ_0 and ϕ_1, where ϕ_0 is the zero map and ϕ_1 is an isomorphism (with a little work, such an embedding can be explicitly constructed without using IV.4.2.2.). Let P_∞ be the inductive limit of the stationary inductive system

$$A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots$$

Then P_∞ is a UCT Kirchberg algebra (I.5.1.46.) with $K_0(P_\infty) = 0$ and $K_1(P_\infty) \cong \mathbb{Z}$. P_∞ can be alternately constructed as a graph algebra (B. Neubüser; cf. [RS02, 4.4.4]).

IV.4.2.6. Lemma. Let A and B be UCT Kirchberg algebras. Then there is a UCT Kirchberg algebra D with $K_0(D) \cong K_0(A) \oplus K_0(B)$ and $K_1(D) \cong K_1(A) \oplus K_1(B)$.

Proof: Construct a unital embedding ϕ of $A \oplus B$ into $A \oplus B$ as follows. Let p and q be nontrivial projections in A and B equivalent to 1_A and 1_B respectively. Then $pAp \cong A$, and the class of $1 - p$ is 0 in $K_0((1-p)A(1-p))$. Thus there is a unital embedding of O_2 into $(1-p)A(1-p)$. Since B embeds unitally in O_2, there is a unital embedding $\psi_B : B \to O_2 \to (1-p)A(1-p)$. Let

$$\phi_A = \psi_A \oplus \psi_B : A \oplus B \to pAp \oplus (1-p)A(1-p) \subseteq A$$

where ψ_A is an isomorphism from A onto pAp. Then ϕ_A is a unital embedding. Similarly construct

$$\phi_B : A \oplus B \to (1-q)B(1-q) \oplus qBq \subseteq B$$

from an isomorphism of B onto qBq and a map $A \to O_2 \to (1-q)B(1-q)$. Set $\phi = \phi_A \oplus \phi_B : A \oplus B \to A \oplus B$. Then ϕ is a unital embedding, and induces an isomorphism from $K_j(A \oplus B) \cong K_j(A) \oplus K_j(B)$ to $K_j(A \oplus B)$ for $j = 0, 1$. Let D be the inductive limit of the stationary inductive system

$$A \oplus B \xrightarrow{\phi} A \oplus B \xrightarrow{\phi} A \oplus B \xrightarrow{\phi} \cdots$$

It is clear that D is in the UCT class and that $K_j(D) \cong K_j(A) \oplus K_j(B)$ $(j = 0, 1)$. We claim D is purely infinite. If $(0,0) \neq (a,b) \in A \oplus B$, then there are $x_A, y_A \in A$ such that $x_A \phi_A(a,b) y_A = 1_A$ and $x_B, y_B \in B$

with $x_B\phi_B(a,b)y_B = 1_B$ since A and B are purely infinite. Thus $(x_A, x_B)\phi(a,b)(y_A, y_B) = 1_{A\oplus B}$. So, if D_∞ is the algebraic direct limit of the sequence, for each nonzero $z \in D_\infty$ there are $x, y \in D_\infty$ with $xzy = 1$, and it follows as in the proof of I.5.1.46. that D is purely infinite. 👍

Using the classification theorem for finitely generated abelian groups, by repeated applications of IV.4.2.6., beginning with the building blocks (first do K_0 and K_1 separately and then put them together with one more application of IV.4.2.6.), we obtain:

IV.4.2.7. LEMMA. Let G_0 and G_1 be finitely generated abelian groups. Then there is a UCT Kirchberg algebra A with $K_0(A) \cong G_0$ and $K_1(A) \cong G_1$.

We then obtain the final result:

IV.4.2.8. THEOREM. Let G_0 and G_1 be countable abelian groups, and $u \in G_0$. Then there is a UCT Kirchberg algebra A with $(K_0(A), K_1(A), [1_A]) \cong (G_0, G_1, u)$.

PROOF: There are increasing sequences $(G_0^{(n)})$ and $(G_1^{(n)})$ where $G_0^{(n)}$ and $G_1^{(n)}$ are finitely generated for all n and $G_0 = \cup_n G_0^{(n)}$, $G_1 = \cup_n G_1^{(n)}$, with $u \in G_0^{(1)}$. For each n, there is a UCT Kirchberg algebra A_n with $(K_0(A_n), K_1(A_n), [1_{A_n}]) \cong (G_0^{(n)}, G_1^{(n)}, u)$ by IV.4.2.7. (the order unit can be arranged as in IV.4.1.1.). By IV.4.2.2., there are unital embeddings $\phi_{n,n+1} : A_n \to A_{n+1}$ inducing the inclusion maps from $G_j^{(n)}$ to $G_j^{(n+1)}$ $(j = 0, 1)$ for each n. Set $A = \lim_\to(A_n, \phi_{n,n+1})$. Then A is UCT and $(K_0(A), K_1(A), [1_A]) \cong (G_0, G_1, u)$, and A is purely infinite by I.5.1.46.. 👍

The proof, combined with the classification theorem, also gives:

IV.4.2.9. COROLLARY. Let A be a UCT Kirchberg algebra. Then A is an inductive limit of a sequence of UCT Kirchberg algebras with finitely generated K-theory.

Chapter V

Classification in the Purely Infinite Case

Chapter VI

Semiprojectivity

The notions of projectivity and semiprojectivity are noncommutative analogs of the topological notions of absolute retract (AR) and absolute neighborhood retract (ANR) respectively. These notions were originally used in the development of shape theory for C*-algebras, first introduced by E. Effros and J. Kaminker [EK86]; semiprojectivity in the modern sense was introduced in [Bla85b]. Semiprojective C*-algebras have rigidity properties which make them conceptually and technically important in several other aspects of C*-algebra theory; this is reflected especially in the work of T. Loring and his coauthors on lifting problems (see, for example, [Lor97]). They are relevant to several aspects of classification of simple C*-algebras, although simple C*-algebras are rarely themselves semiprojective. It is not too easy for a C*-algebra to be semiprojective, but there does seem to be a reasonable supply of such algebras.

In this chapter, all C*-algebras will be *separable* (unless otherwise specified or constructed), although some of the definitions and results make sense more generally. We will consider several categories: $\mathcal{S}$, the category of separable C*-algebras and *-homomorphisms (our primary category); $\mathcal{S}_1$, the category of separable unital C*-algebras and unital *-homomorphisms; $\mathcal{C}$, the category of separable commutative C*-algebras and *-homomorphisms; and $\mathcal{C}_1$, the category of unital commutative C*-algebras and unital *-homomorphisms. Recall () that $\mathcal{C}_1$ is contravariantly isomorphic to the category of compact metrizable spaces and continuous functions.

VI.1. Absolute Retracts and Absolute Neighborhood Retracts

The theory of absolute retracts and absolute neighborhood retracts is a vast and important part of topology. We will make no attempt to give a comprehensive treatment of the subject, but simply describe the basic aspects, concentrating on those which are relevant to the generalization to C*-algebras. In particular, we will only consider the case of compact metrizable spaces (that is, we will work in the category $\mathcal{C}_1$, regarding it as the category of compact metrizable spaces and continuous functions). See e.g. [Bor67] or [Hu65] for a more comprehensive treatment of the topological theory.

In this section, "space" will mean "compact metrizable space" unless otherwise specified, and "map" will mean "continuous function."

VI.1.1. Absolute Retracts

VI.1.1.1. Definition. Let Y be a topological space, and Z a subset of Y. A *retraction* of Y onto Z is a map $r : Y \to Z$ such that $r(z) = z$ for all $z \in Z$. In other words, r is a left inverse for the inclusion map $i : Z \to Y$. Z is a *retract* of Y if there exists a retraction of Y onto Z.

A *deformation retraction* from Y to Z is a map $f : [0,1] \times Y \to Y$ such that

(i) $f(0, y) = y$ for all $y \in Y$.

(ii) $f(1, y) \in Z$ for all $y \in Y$.

(iii) $f(1, z) = z$ for all $z \in Z$.

A deformation retraction is a *strong deformation retraction* if it satisfies the stronger condition

(iii′) $f(t, z) = z$ for all $z \in Z$ and all $t \in [0,1]$.

If f is a deformation retraction from Y onto Z, then $r : Y \to Z$ defined by $r(y) = f(1, y)$ is a retraction from Y onto Z. There is no converse: if Z is a retract of Y, there is not in general a deformation retraction of Y onto Z (a singleton subset is a retract!) And if there is a deformation retraction, there is not in general a strong deformation retraction (VI.1.1.13.(iv)). If Y is a space by our definition (i.e. compact and metrizable) and Z is a retract of Y, then Z is necessarily compact and thus closed in Y (more generally, a retract of a Hausdorff space is necessarily closed).

VI.1.1.2. Definition. A (compact metrizable) space X is an *absolute retract (AR)* if, whenever Y is a space and ϕ is a homeomorphism from X onto a (necessarily closed) subspace of Y, then there is a retraction from Y onto $\phi(X)$.

In other words, X is an absolute retract if it is a retract of any space in which it is embedded.

The Tietze Extension Property

VI.1.1.3. Definition. A space X has the *Tietze Extension Property* if, whenever Y is a space and Z a closed subspace of Y, any map $g : Z \to X$ extends to a map $f : Y \to X$. A space with the Tietze Extension Property is called an *absolute extensor* or AE.

VI.1.1.4. The Tietze Extension Theorem says that $[0,1]$ has the Tietze Extension Property. By working coordinatewise, any product of intervals has the Tietze Extension Property. In particular, the Hilbert cube $[0,1]^{\mathbb{N}}$ has the Tietze Extension Property.

VI.1.1.5. Proposition. Let X be a space and K a retract of X. If X has the Tietze Extension Property, then so does K.

Proof: Let r be a retraction from X onto K. Let Y be a space, Z a closed subspace, and $g : Z \to K$ a map. Regard g as a map from Z to X, and extend g to a map $h : Y \to X$. Then $f = r \circ h$ is an extension of g to a map from Y to K. 👍

The next result gives an important alternate characterization of absolute retracts:

VI.1.1.6. PROPOSITION. Let X be a space. Then X has the Tietze Extension Property (i.e. is an AE) if and only if it is an absolute retract.

PROOF: Suppose X has the Tietze Extension Property, and ϕ is a homeomorphism from X onto a subspace of Y. The map $\phi^{-1} : \phi(X) \to X$ extends to a map $f : Y \to X$. Then $r = \phi \circ f$ is a retraction from Y onto $\phi(X)$.

Conversely, suppose X is an AR. Then X embeds in the Hilbert cube, as does any space, and is a retract of the Hilbert cube. Since the Hilbert cube has the Tietze Extension Property, so does X by VI.1.1.5.. 👍

The next corollary also has a straightforward direct proof.

VI.1.1.7. COROLLARY. Any retract of an AR is an AR.

Contractibility

VI.1.1.8. DEFINITION. Let X be a topological space. A *contraction* of X is a deformation retraction from X onto $\{p\}$, for some $p \in X$ (contraction to p), i.e. a function $f : [0,1] \times X \to X$ satisfying

(i) $f(0, x) = x$ for all $x \in X$.

(ii) $f(1, x) = p$ for all $x \in X$.

A contraction to p is a *strong contraction to* p if it is a strong deformation retraction to $\{p\}$, i.e. satisfies

(iii′) $f(t, p) = p$ for all $t \in [0, 1]$.

X is *contractible* if it there is a contraction to some $p \in X$. If $p \in X$, X is *strongly contractible to* p if there is a strong contraction of X to p. X is *strongly contractible* if it is strongly contractible to p for every $p \in X$.

VI.1.1.9. A contractible space is path-connected; thus if a space is contractible to one point, it is contractible to any point. But this does not hold for strong contractibility: a space may be strongly contractible to one point but not another (VI.1.1.13.(iv)). In particular, a contractible space is not necessarily strongly contractible.

VI.1.1.10. PROPOSITION. An AR is strongly contractible.

PROOF: Let X be an AR, and $p \in X$. Let Y be the cone over X, i.e. the quotient $([0,1] \times X)/(\{1\} \times X)$ obtained by collapsing $\{1\} \times X$ to a point. Let Z be the closed subspace of Y consisting of the union of the base $\{0\} \times X$ and the spike $[0,1] \times \{p\}$. Let $g : Z \to X$ be the evident map: $g(0, x) = x$ and $g(t, p) = p$ for all x and t. Extend g to $f : Y \to X$. Then f defines a strong retraction of X to p. 👍

The converse of VI.1.1.10. appears to be open.

VI.1.1.11. Thus any map of a space Y to an AR X is homotopic to a constant map, and in particular any two maps of Y to X are homotopic. A stronger statement is true:

VI.1.1.12. Proposition. Let X be an AR, Y a space, and Z a closed subset of Y. Let g_0 and g_1 be maps from Y to X such that $g_0|_Z = g_1|_Z$. Then g_0 and g_1 are homotopic rel Z, i.e. there is a map $f : [0,1] \times Y \to X$ such that $f(0,y) = g_0(y)$ and $f(1,y) = g_1(y)$ for all $y \in Y$ and $f(t,z) = g_0(z)(= g_1(z))$ for all $t \in [0,1]$ and $z \in Z$.

Proof: Let $A = (\{0\} \times Y) \cup (\{1\} \times Y) \cup ([0,1] \times Z)$. Then A is a closed subset of $[0,1] \times Y$, and g_0 and g_1 define an obvious map from A to X, whose extension to $[0,1] \times Y$ gives f.

VI.1.1.13. Examples. (i) $[0,1]$, $[0,1]^n$, $[0,1]^{\mathbb{N}}$ are absolute retracts. A single point is an AR.

(ii) Any (compact) retract of $\mathbb{R}^n$ is a retract of $[0,1]^n$, hence an AR. For example, any finite tree is a retract of $\mathbb{R}^2$, hence an AR.

(iii) More generally, a dendrite (connected compact metrizable space in which any two distinct points can be separated by a third point) is an AR; in fact these are precisely the one-dimensional ARs. In particular, any space of the form in Figure VI.1 with any (finite) number of whiskers at each node is an AR. There are $2^{\aleph_0}$ mutually nonhomeomorphic such spaces, so there are $2^{\aleph_0}$ topologically distinct 1-dimensional ARs.

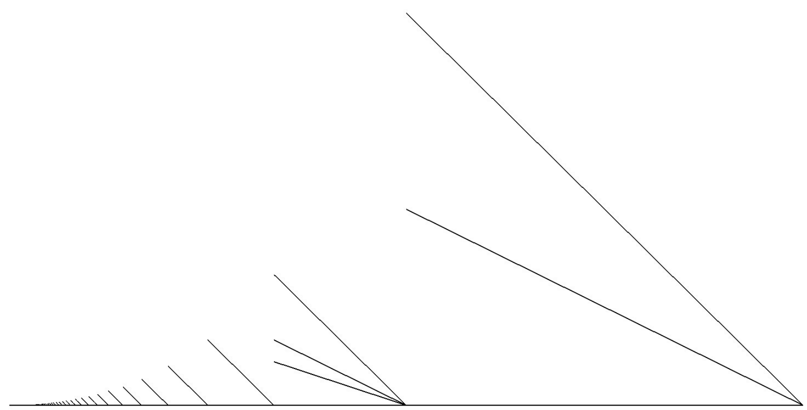


Figure VI.1: A dendrite

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A dendrite with a more complicated local structure is the *rational ruler*: to the interval $[0,1]$ on the x-axis in $\mathbb{R}^2$ add the vertical closed interval between $(r,0)$ and $\left(r, \frac{1}{q}\right)$ whenever $r = \frac{p}{q}$ is a rational number in lowest terms in $[0,1]$ (with $0 = \frac{0}{1}$).

(iv) Let X be the *infinite comb* in $\mathbb{R}^2$, consisting of the closed horizontal interval between $(0,0)$ and $(1,0)$, the vertical line segment between $(0,0)$ and $(0,1)$, and the vertical line segments between $\left(\frac{1}{n},0\right)$ and $\left(\frac{1}{n},1\right)$ for all $n \in \mathbb{N}$ (Figure VI.2). Then X is contractible, but not locally contractible to any nonzero point p on the y-axis since any path in X from p to a point of X in the open right half plane must go through the origin. In particular, X is not strongly contractible, hence not an AR.

(v) A circle is not an AR since it is not contractible. More generally, any graph containing a cycle is not an AR.

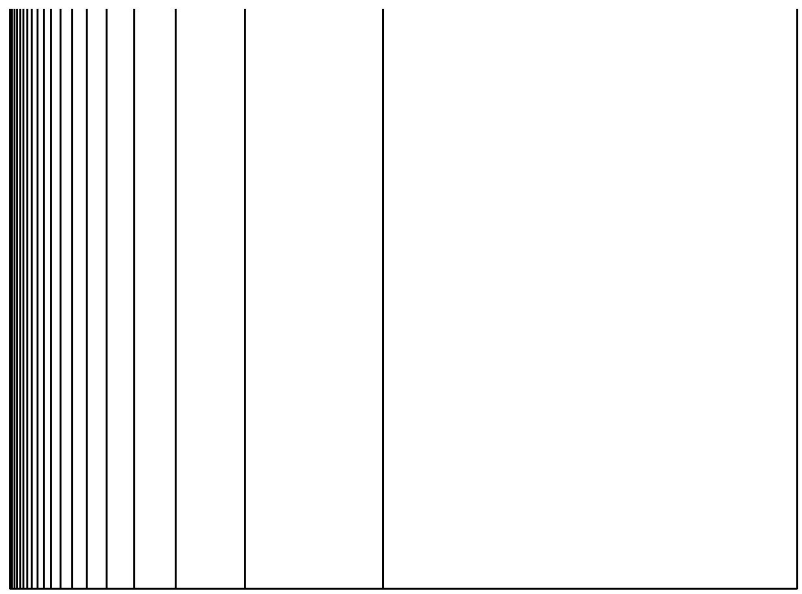

Figure VI.2: The infinite comb

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(vi) There are many higher-dimensional ARs. For example, a polyhedron is an AR if and only if it is contractible. And it follows from VI.1.1.6. that an arbitrary (finite or countable) product of ARs is an AR.

(vii) A number of pathological examples can be found in [Bor67].

VI.1.2. Absolute Neighborhood Retracts

For C*-algebra purposes, it is more important to consider a "local" version of absolute retracts:

VI.1.2.1. Definition. Let X be a space. X is an *absolute neighborhood retract (ANR)* if, whenever Y is a space and ϕ a homeomorphism of X onto a subspace of Y, there is an open neighborhood U of $\phi(X)$ and a retraction from U onto $\phi(X)$.

Of course, every AR is an ANR. But ANRs are much more abundant than ARs, as we shall see.

There is an analogy of the Tietze Extension Property:

VI.1.2.2. Proposition. Let X be a space. Then X is an ANR if and only if, whenever Y is a space and Z a closed subspace of Y, and g is a map from Z to X, there is an open neighborhood U of Z in Y such that g extends to a map $f : U \to X$ (a space with this property is called an *absolute neighborhood extensor* or ANE).

Proof: The proof is very similar to the proof of VI.1.1.6.. Suppose X is an ANE, and ϕ is a homeomorphism from X onto a subspace of Y. The map $\phi^{-1} : \phi(X) \to X$ extends to a map $f : U \to X$ for some open neighborhood U of $\phi(X)$. Then $r = \phi \circ f$ is a retraction from U onto $\phi(X)$.

Conversely, suppose X is an ANR. Then X embeds in $[0,1]^{\mathbb{N}}$, and there is a neighborhood V of X in $[0,1]^{\mathbb{N}}$ and a retraction r from V onto X. Let Y be a space and Z a closed subspace of Y, and let $g : Z \to X$ be a map. Regard g as a map from Z to $[0,1]^{\mathbb{N}}$, and let h be an extension to Y. Set $U = h^{-1}(V)$. Then U is an open neighborhood of Z and $f = r \circ h|_U$ is an extension of g to U. 👍

The second half of the proof begins by noting that every ANR is a retract of an open set in the Hilbert cube.

A retract of an ANR is an ANR. More generally:

VI.1.2.3. Proposition. A compact retract of an open set in an ANR is an ANR.

Proof: Let X be an ANR, V an open set in X, and r a retraction of V onto a compact subset K of V. Let Y be a space, Z a closed subspace of Y, and $g : Z \to K$ a map. Regard g as a map to X, and extend g to a map h from an open set W containing Z to X. Set $U = h^{-1}(V)$; then U is an open set in Y containing Z and contained in U. Then $f = r \circ h|_U$ maps U to K and extends g. 👍

A relevant property is local contractibility. This can be defined in more than one inequivalent way; we give what seems to be the standard definition, which is also the most relevant one for our purposes.

VI.1.2.4. Definition. Let X be a topological space. If $p \in X$, then X is *locally contractible to* p if for every open neighborhood U of p there is an open neighborhood V of p contained in U which is contractible to p within U. X is *locally contractible* if it is locally contractible to p for any $p \in X$.

If $p \in X$, then X is *locally strongly contractible to* p if for every open neighborhood U of p there is an open neighborhood V of p contained in U which is strongly contractible to p within U. X is *locally strongly contractible* if it is locally contractible to p for any $p \in X$.

VI.1.2.5. A locally contractible space need not be contractible. For example, every topological manifold and every polyhedron (underlying space of a finite simplicial complex) is locally contractible. Conversely, a contractible space need not be locally contractible (), and a locally contractible space need not be locally strongly contractible.

The next result has a straightforward proof:

VI.1.2.6. Proposition. A strongly contractible space is locally strongly contractible.

VI.1.2.7. PROPOSITION. Every ANR is locally strongly contractible.

PROOF: Let X be an ANR. Embed X in the Hilbert cube, and let U be an open neighborhood of X in the Hilbert cube and r a retraction from U onto X. Let $p \in X$. U is locally strongly contractible; let $f : [0,1] \times V \to V$ be a strong local contraction of an open set V contained in U to p within U. 👍

VI.1.2.8. Local strong contractions do not seem to have been much studied in the theory of ANRs; many results concern local contractibility but few if any mention local strong contractibility. It is known that every locally contractible space of finite topological dimension is an ANR (it follows that every compact topological manifold with or without boundary is an ANR), but this can fail for an infinite-dimensional space; see [Bor67] for discussion and proof. But I conjecture:

VI.1.2.9. CONJECTURE. A (compact metrizable) space is an ANR if and only if it is locally strongly contractible.

To the best of my knowledge, no counterexample is known. (Borsuk's example of an infinite-dimensional space which is locally contractible but not an ANR is not locally strongly contractible.)

Although an ANR is not necessarily connected (a finite separated union of ANRs is an ANR), a simple consequence of VI.1.2.7. is:

VI.1.2.10. COROLLARY. An ANR has only finitely many connected components, all of which are path-connected and locally path-connected.

VI.1.3. Locally Compact and Metrizable ANRs

We need to briefly discuss ARs and ANRs in the broader context of metrizable spaces; the locally compact case will be the only one of some importance to us. We denote by $\mathcal{M}$ the class of metrizable spaces. For more details, and proofs of the results, see e.g. [Bor67].

VI.1.3.1. DEFINITION. A metrizable space X is an *AR($\mathcal{M}$)* if, whenever Y is a metrizable space and ϕ is a homeomorphism from X onto a closed subspace of Y, then there is a retraction from Y onto $\phi(X)$.

A metrizable space X is an *ANR($\mathcal{M}$)* if, whenever Y is a metrizable space and ϕ a homeomorphism of X onto a closed subspace of Y, there is an open neighborhood U of $\phi(X)$ and a retraction from U onto $\phi(X)$.

The definitions are formally the same as in the compact case, made within the category $\mathcal{M}$, with the exception that it is not automatic that the image $\phi(X)$ is closed in Y and must be specifically assumed.

VI.1.3.2. Many of the results in the compact case carry over to the metrizable case, including the characterization by the Tietze Extension Property, the Homotopy Extension Property (), and the fact that a retract of an AR($\mathcal{M}$) [resp. an ANR($\mathcal{M}$)] is an AR($\mathcal{M}$) [resp. an ANR($\mathcal{M}$)]. An ANR($\mathcal{M}$) is an AR($\mathcal{M}$) if and only if it is contractible. There is also a more general version of VI.1.2.3.:

VI.1.3.3. Theorem. An open set in an ANR($\mathcal{M}$) is an ANR($\mathcal{M}$).

And we have:

VI.1.3.4. Theorem. Let X be a metrizable space. If X is a finite or countable union of open sets, each of which is an ANR($\mathcal{M}$), then X is an ANR($\mathcal{M}$).

VI.1.3.5. Corollary. Let X be a separable metrizable space. Then X is an ANR($\mathcal{M}$) if and only if each point of X has an open neighborhood which is an ANR($\mathcal{M}$).

VI.1.3.6. One might think this result could be combined with the local contractibility of ANR($\mathcal{M}$)'s to conclude that an ANR($\mathcal{M}$) is "locally an AR($\mathcal{M}$)" (i.e. every point has a neighborhood which is an AR($\mathcal{M}$)), but there is a subtle difficulty: local contractibility (according to the standard definition) does not mean every point has a contractible neighborhood. In fact, there exist compact ANR($\mathcal{M}$)'s (i.e. ANRs) in which points do not have compact neighborhoods which are ARs [Bor67, p. 152-156].

A compact AR($\mathcal{M}$) [resp. ANR($\mathcal{M}$)] is obviously an AR [resp. an ANR]. The converse also holds:

VI.1.3.7. Theorem. Let X be a compact metrizable space. Then X is an AR [resp. an ANR] if and only if it is an AR($\mathcal{M}$) [resp. an ANR($\mathcal{M}$)].

This is proved by noting that a compact metrizable ANR is a retract of an open set in the Hilbert cube, which is an ANR($\mathcal{M}$) by the full form of the Tietze Extension Theorem.

VI.1.3.8. It is then natural to define a *locally compact AR* [resp. a *locally compact ANR*] to be a locally compact metrizable space which is an AR($\mathcal{M}$) [resp. an ANR($\mathcal{M}$)]. In fact, any other reasonable definition is equivalent to this one (however, see ()).

VI.1.4. Translatable and Untranslatable Results

Our goal is to translate the definitions and results about ARs and ANRs into C*-algebra language in a way that makes sense and gives correct results. Most things about absolute retracts translate nicely, but the situation for ANRs is more complicated: some things can be nicely translated, others not.

VI.1.4.1. Since the category of compact metrizable spaces is isomorphic to the category $\mathcal{C}_1$, in principle everything about ARs and ANRs (and everything else about spaces) can be phrased in C*-algebra language. But it is clumsy or tricky to phrase some things in a useful way, e.g. open neighborhoods and open covers, local contractibility, finite-dimensionality. And some definitions and results for ANRs do not hold or even make sense for noncommutative C*-algebras.

VI.1.4.2. Since the correspondence between spaces and C*-algebras is contravariant, translating statements involves "turning arrows around," and replacing subobjects by quotients and vice versa. Problems of extending continuous functions on the space level, one of the fundamental points of the theory of ANRs, become problems of lifting homomorphisms into quotients.

VI.1.4.3. Statements and results about ANRs fall roughly into three classes:

(i) Things whose statement and proof translate nicely in a straightforward way to results valid also in the noncommutative case.

(ii) Things whose statement translates nicely to a result which holds in the noncommutative case, but with a completely different proof required.

(iii) Things which do not hold or even make sense in the noncommutative setting.

We will give examples of each type.

An Alternate Characterization of ANRs

Even the definition of ANRs does not translate well into C*-algebra language, much less have a direct noncommutative version. Here is an alternate characterization of ANRs, which is the key to making a reasonable noncommutative generalization:

VI.1.4.4. Proposition. Let X be a space. Then X is an ANR if and only if, whenever Y is a space, (Z_n) a decreasing sequence of closed subspaces of Y with $Z = \cap_{n=1}^{\infty} Z_n$, and $g : Z \to X$ is a map, then there is an extension f_n of g to some Z_n (hence to Z_n for all sufficiently large n).

Proof: Suppose X is an ANR and Y, (Z_n), and g are given. Then g extends to $f : U \to X$ for some open neighborhood U of Z. Since $Z \subseteq U$, the open sets Z_n^c form an open cover of U^c which has a finite subcover; since the sets are nested, we have $U^c \subseteq Z_n^c$ for some n, i.e. $Z_n \subseteq U$ for some n (all sufficiently large n).

Conversely, suppose X satisfies the condition in the statement, Y is a space, Z a closed subspace, and $g : Z \to X$ a map. Since Y is second countable and normal, Z is a G_δ in Y, and there is a decreasing sequence (U_n) of open sets in Y with $\bar{U}_{n+1} \subseteq U_n$ for each n and $Z = \cap_{n=1}^{\infty} U_n$. If $Z_n = \bar{U}_n$ for each n, then (Z_n) is a decreasing sequence of compact sets with $Z = \cap_{n=1}^{\infty} Z_n$, so by assumption g extends to some Z_n, i.e. to some U_n.

VI.1.4.5. **Caution:** This characterization only works in the compact case. For example, a countable discrete space is a locally compact ANR in the sense of (), but does not satisfy this condition.

VI.2. Semiprojectivity

VI.2.1. Definition of Projective and Semiprojective C*-Algebras

We take the definitions of AR and ANR (in the form of VI.1.4.4.) and turn the arrows around in a straightforward way to obtain our definitions of projectivity and semiprojectivity. In the following, $\mathcal{B}$ will denote one of the categories $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{C}$, $\mathcal{C}_1$ (the definition could be applied to other categories of C*-algebras).

VI.2.1.1. Definition. A C*-algebra A in $\mathcal{B}$ is *semiprojective in* $\mathcal{B}$ if, for any C*-algebra B in $\mathcal{B}$, increasing sequence $\langle J_n \rangle$ of (closed two-sided) ideals of B, with $J = [\cup J_n]^-$ (we say such B, (J_n), J form a *tower*), and $\mathcal{B}$-morphism $\phi : A \to B/J$, there is an n and a $\mathcal{B}$-morphism $\psi : A \to B/J_n$ such that $\phi = \pi \circ \psi$, where $\pi : B/J_n \to B/J$ is the natural quotient map.

A ϕ for which such a ψ exists is said to be *partially liftable*. If there is a $\psi : A \to B$ with $\phi = \pi \circ \psi$, then ϕ is *liftable*; if every homomorphism from A is liftable, A is said to be *projective*.

The terms "projective" and "semiprojective" without further qualification will mean "projective in $\mathcal{S}$" and "semiprojective in $\mathcal{S}$" respectively.

VI.2.1.2. The definition of a projective C*-algebra is standard from category theory: a C*-algebra which is projective in $\mathcal{B}$ is precisely a projective object in the category $\mathcal{B}$. Pictorially:

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[ur, dotted, "\psi"] \arrow[r, "\phi"'] & B/J \end{tikzcd}$$

VI.2.1.3. There is a corresponding diagram for semiprojectivity:

$$\begin{tikzcd} & B \arrow[d] \\ & B/J_1 \arrow[d] \\ & \vdots \arrow[d] \\ & B/J_n \arrow[d] \\ & \vdots \arrow[d] \\ A \arrow[uur, dotted, "\psi"] \arrow[r, "\phi"'] & B/J \end{tikzcd}$$

So one can partially lift to within a finite distance from the top, but not necessarily all the way.

VI.2.1.4. **Mea culpa:** The term "semiprojective" was first defined in [EK86] to mean something different (less restrictive). The author redefined it in the form of VI.2.1.1. in [Bla85b]. Although subsequent developments have amply justified that the author's definition is the "right" noncommutative version of ANRs, it was a mistake to reuse the term "semiprojective"; redefining it led to some apparent hard feelings which I did not intend and regret. My advice to any other author who may be in a similar position: use

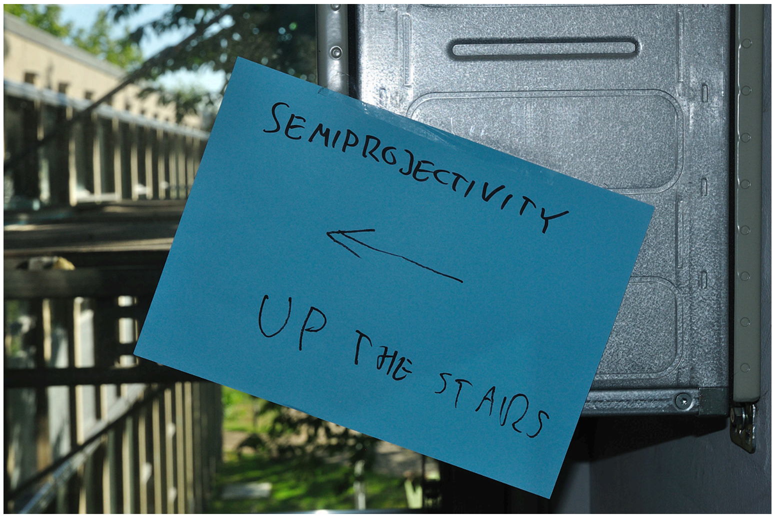

Figure VI.3: But not necessarily all the way to the top. Copenhagen Semiprojectivity Workshop, 2010.
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a new term instead of redefining an existing term (although I don't know what other term I should have chosen in this case).

This definition was specifically designed to make the following true:

VI.2.1.5. Proposition. Let X be a (compact metrizable) space. Then

(i) $C(X)$ is projective in $\mathcal{C}_1$ if and only if X is an AR.

(ii) $C(X)$ is semiprojective in $\mathcal{C}_1$ if and only if X is an ANR.

VI.2.1.6. **Caution:** These results hold only in the compact case, not for locally compact ARs and ANRs.

VI.2.1.7. If A is [semi]projective in $\mathcal{S}$ or $\mathcal{S}_1$ and is in a full subcategory $\mathcal{B}$, then it is [semi]projective in $\mathcal{B}$ also. But the converse is not true in general: if X is an ANR, then $C(X)$ is semiprojective in $\mathcal{C}_1$, but not necessarily semiprojective in $\mathcal{S}_1$ (VI.2.5.2.). (It turns out that semiprojectivity in $\mathcal{S}$ and $\mathcal{S}_1$ are equivalent, but this is false for projectivity.)

VI.2.1.8. For convenience, we have only defined semiprojectivity for separable C*-algebras (although the same definition makes sense also for nonseparable C*-algebras, it is probably not the appropriate one). Thus all semiprojective C*-algebras will implicitly be separable. In the definition, it does not matter whether B is required to be separable (the utility of allowing nonseparable B is illustrated in VI.2.5.2.). This is because, when given A, B, (J_n), and ϕ, to obtain a partial lifting of ϕ, B can be replaced by any C*-subalgebra D such that $\pi(D)$ contains $\phi(A)$. (There is a subtlety in showing this, however: one needs to know that $\cup_n(D \cap J_n)$ is dense in $D \cap J$, which follows from [Bla06, II.5.1.3].) Thus the definition of semiprojectivity does not change if in VI.2.1.1. we make any or all of the following restrictions:

(i) B is separable.

(ii) ϕ is surjective.

(iii) ϕ is injective.

(For (i) and (ii), replace B by the C*-subalgebra generated by a set of preimages of a countable dense set in $\phi(A)$; for (iii), replace B by $A \oplus B$ and J_n by $0 \oplus J_n$.)

VI.2.2. Stable Relations

Presentations of C*-Algebras

Many important C*-algebras can be simply and naturally expressed as universal C*-algebras on sets of generators and relations. This subsection is partially excerpted from [Bla06].

VI.2.2.1. Suppose a set $\mathcal{G} = \{x_i : i \in \Omega\}$ of generators and a set $\mathcal{R}$ of relations are given. The relations can be of a very general nature, but are usually algebraic relations among the generators and their adjoints, or more generally of the form

$$\|p(x_{i_1}, \cdots, x_{i_n}, x_{i_1}^*, \cdots, x_{i_n}^*)\| \leq \eta$$

where p is a polynomial in $2n$ noncommuting variables with complex coefficients and $\eta \geq 0$. The only restriction on the relations is that they must be realizable among operators on a Hilbert space and they must (at least implicitly) place an upper bound on the norm of each generator when realized as an operator. A *representation* of $(\mathcal{G}|\mathcal{R})$ has the obvious meaning: a set $\{T_i : i \in \Omega\}$ of bounded operators on a Hilbert space $\mathcal{H}$ satisfying the relations. A representation of $(\mathcal{G}|\mathcal{R})$ defines a *-representation of the free *-algebra $\mathcal{A}$ on the set $\mathcal{G}$. For $x \in \mathcal{A}$, let

$$\|x\| = \sup\{\|\pi(x)\| : \pi \text{ a representation of } (\mathcal{G}|\mathcal{R})\}.$$

If this supremum is finite for all $x \in \mathcal{A}$ (it is enough to check this on the generators), it defines a C*-seminorm on $\mathcal{A}$, and the completion (with elements of seminorm 0 divided out) is called the *universal C*-algebra on* $(\mathcal{G}|\mathcal{R})$, denoted $C^*(\mathcal{G}|\mathcal{R})$. We may also consider the universal unital C*-algebra on a set of generators and relations by adding an additional generator 1 with relations $1 = 1^* = 1^2$ and $x1 = 1x = x$ for each other generator.

This construction is extremely general; indeed, every C*-algebra can be written (in an uninteresting way) as the universal C*-algebra on a suitable set of generators and relations. A more precise (but generally still rather uninteresting) statement is true in the separable case:

VI.2.2.2. PROPOSITION. Let A be a separable C*-algebra. Then $A \cong C^*(\mathcal{G}|\mathcal{R})$ for a countable set $\mathcal{G} = \{g_1, g_2, \dots\}$ with $\|g_n\| \to 0$ and a countable set $\mathcal{R}$ of relations. Conversely, any such universal C*-algebra is separable.

Proof: A contains a countable dense *-subring A_0 which is an algebra over the (countable) field $\mathbb{F} = \mathbb{Q}+\mathbb{Q}i$. The set of sum, product, adjoint, and scalar multiple relations is countable and determines A_0 as a *-algebra over $\mathbb{F}$. If $\{x_n : n \in \mathbb{N}\}$ is an enumeration of the elements of A_0, we take the set of all these algebraic relations and one additional relation $\|x_n\| \leq \|x_n\|_A$ for each n; then clearly A is the universal C*-algebra on these generators and relations. To satisfy the last condition, set $g_n = \frac{x_n}{n\|x_n\|_A}$ for each n. Any algebraic relation in the x_n can be converted to a relation in the g_n by introducing scalar coefficients. 👍

VI.2.2.3. Fixing a set of generators $\{g_n : n \in \mathbb{N}\}$ for a separable C*-algebra A with $\|g_n\| \to 0$ is a rough noncommutative analog of choosing a metric for a compact metrizable space, allowing "uniform" arguments to be made.

VI.2.2.4. If $A \cong C^*(\mathcal{G}|\mathcal{R})$, then $(\mathcal{G}|\mathcal{R})$ is a *presentation* of A. A C*-algebra generally has many wildly different presentations (and it is generally hopelessly difficult to say when two presentations give isomorphic C*-algebras). The most interesting presentations are finite ones, and where the relations are well-behaved ().

VI.2.2.5. Examples.

(i) There is no "universal C*-algebra generated by a single self-adjoint element," since there is no bound on the norm of the element. But there is a "universal C*-algebra generated by a single self-adjoint element of norm one," with $\mathcal{G} = \{x\}$ and $\mathcal{R} = \{x = x^*, \|x\| \leq 1\}$. This C*-algebra is isomorphic to $C_0([-1,1])$, the continuous functions on $[-1,1]$ vanishing at zero, via functional calculus. Similarly, the "universal unital C*-algebra generated by a self-adjoint element of norm one" is isomorphic to $C([-1,1])$. There is a "universal C*-algebra generated by a single positive element of norm one," isomorphic to $C_0((0,1])$, and a "universal C*-algebra generated by a single normal element of norm one," isomorphic to $C_0(\bar{\mathbb{D}})$, where $\bar{\mathbb{D}}$ is the closed unit disk in $\mathbb{C}$.

(ii) There is a "universal C*-algebra generated by a single unitary," since a unitary automatically has norm one. We may take $\mathcal{G} = \{u, 1\}$,

$$\mathcal{R} = \{1 = 1^* = 1^2, u1 = 1u = u, u^*u = uu^* = 1\}.$$

$C^*(\mathcal{G}|\mathcal{R}) \cong C(\mathbb{T})$, the continuous functions on the circle. Similarly, the universal C*-algebra generated by n commuting unitaries is isomorphic to $C(\mathbb{T}^n)$, continuous functions on the n-torus.

(iii) If G is a group, there is a "universal C*-algebra generated by a group of unitaries isomorphic to G," with generators and relations from any presentation of G and additional relations making each generator a unitary. The corresponding C*-algebra is called the *group C*-algebra* of (discrete) G, denoted $C^*(G)$. The representations of $C^*(G)$ are in canonical one-one correspondence with the unitary representations of G. For example, $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ and $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$. Another important example is $\mathbb{F}_n$, the free group on n generators; $C^*(\mathbb{F}_n)$ is the universal C*-algebra generated by n unitaries. To see that the group G actually embeds in the unitary group of $C^*(G)$, consider the *left regular representation* λ of G on $l^2(G)$ induced by G acting on itself by left translation. The representation λ extends to a representation of $C^*(G)$, which is not faithful in general; the quotient $\lambda(C^*(G))$ is called the *reduced group C*-algebra* of G, denoted $C^*_r(G)$. The group C*-algebra construction generalizes to locally compact topological groups.

(iv) Let $\mathcal{G} = \{e_{ij} : 1 \le i,j \le n\}$,

$$\mathcal{R} = \{e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il} : 1 \le i,j,k,l \le n\},$$

where δ_{jk} is the Kronecker symbol. The relations imply that each generator is a partial isometry, hence of norm 1, so the universal C*-algebra exists. $C^*(\mathcal{G},\mathcal{R}) \cong \mathbb{M}_n$. A set of nonzero elements $\{f_{ij}\}$ in a C*-algebra A satisfying the relations is called a *set of matrix units* of type $\mathbb{M}_n$ in A, and generate a (possibly nonunital) C*-subalgebra isomorphic to $\mathbb{M}_n$. There is a similar description of $\mathbb{K}$ as the universal C*-algebra generated by an (infinite) set of matrix units.

More generally, if D is a finite-dimensional C*-algebra, then D is isomorphic to a direct sum $\oplus_{r=1}^m \mathbb{M}_{n_r}$ of matrix algebras. The natural matrix units

$$\{e_{ij}^{(r)} : 1 \le r \le m, 1 \le i,j \le n_r\}$$

satisfy the relations $e_{ij}^{(r)*} = e_{ji}^{(r)}, e_{ij}^{(r)}e_{kl}^{(s)} = \delta_{rs}\delta_{jk}e_{il}$ for all i,j,k,l,r,s. Conversely, the universal C*-algebra generated by generators satisfying these relations is isomorphic to D. Such a set of generators is called a *set of matrix units of type D*.

(v) There is a "universal C*-algebra generated by a single isometry," the *Toeplitz algebra* () and denoted $\mathcal{T}$. If v is the generator, then $\mathcal{T}$ has an identity v^*v.

In addition to these basic examples, here are some more classes which play a very important role in the theory:

VI.2.2.6. Examples.

(i) Let θ be a real number (usually in the unit interval), and let A_θ be the universal C*-algebra generated by two unitaries u, v, with the relation $vu = e^{2\pi i\theta}uv$. A_θ is called a *rotation algebra*, and in the particularly important case where θ is irrational, an *irrational rotation algebra* (). An irrational rotation algebra is simple. Rotation algebras can be alternately described as crossed products (**??**). More generally, a *noncommutative torus* is a universal C*-algebra generated by unitaries $u_1, \cdots, u_n$, with $u_ku_j = e^{2\pi i\theta_{jk}}u_ju_k$ for all j,k, where $\theta_{jk} \in \mathbb{R}$ (there are necessary relations among the θ_{jk} in order to get a nonzero algebra). The C*-algebra is simple if the θ_{jk} are sufficiently irrational (specifically, if for each j there is a k with θ_{jk} irrational). Noncommutative tori arise in applications (e.g. the quantum Hall effect in physics [Con94], and string theory [**?**]) and are a natural setting for noncommutative geometry. The simple noncommutative tori all turn out to be A$\mathbb{T}$ algebras and thus in the classifiable class.

(ii) Set $\mathcal{G} = \{s_1, \cdots, s_n\}$,

$$\mathcal{R} = \left\{ s_i^*s_i = 1, \sum_{j=1}^n s_js_j^* = 1 : 1 \le i \le n \right\}.$$

$C^*(\mathcal{G}|\mathcal{R})$ is the *Cuntz algebra O_n* (), the universal (unital) C*-algebra generated by n isometries whose range projections are mutually orthogonal and add up to the identity. O_n is purely infinite (), hence simple. There is also O_∞ (), the universal C*-algebra generated by a sequence of isometries with mutually orthogonal range projections; O_∞ is also purely infinite and hence simple. The simplicity of these algebras means that any set of isometries satisfying the relations generates a C*-algebra isomorphic to O_n.

(iii) As a generalization of (ii), we have the Cuntz-Krieger algebras (): let A be an $n \times n$ matrix of zeroes and ones, and let O_A be the universal C*-algebra generated by partial isometries $s_1, \cdots, s_n$ with relations

$$s_i^* s_i = \sum_{j=1}^{n} A_{ij} s_j s_j^*$$

and with $s_i s_i^* \perp s_j s_j^*$ for $i \neq j$ (the last relations are often automatic from the first ones). The range projections of the s_i add up to an identity, so O_A is unital. If A has all ones, $O_A \cong O_n$. O_A is simple (and purely infinite) if and only if A is irreducible (the corresponding directed graph is connected) and not a permutation matrix.

More generally, if A is a row-finite infinite matrix of zeroes and ones (every row has only finitely many ones), corresponding to a directed graph in which every vertex is the source of only finitely many edges, one can form a universal C*-algebra O_A as above. If A is infinite, O_A is nonunital. Such graph C*-algebras have been extensively studied; see, for example, [**?**].

VI.2.2.7. There is a question whether polynomials with constant term should be allowed as relations in the general construction of universal C*-algebras (as opposed to universal unital C*-algebras). There is no harm in allowing polynomials with constant term; in a representation, such a constant term is interpreted as the corresponding multiple of the identity.

The only subtlety is that a polynomial with constant term may not be realizable among bounded operators (on a nonzero Hilbert space). The most famous example is the Canonical Anticommutation Relation from quantum mechanics $p(x, y) = xy - yx - 1$. [Suppose x and y satisfy this relation. If $\lambda \in \sigma(xy)$, $\lambda \neq 0$, then $\lambda \in \sigma(yx)$ by (), so $\lambda + 1 \in \sigma(yx + 1) = \sigma(xy)$; if $\lambda \in \sigma(xy)$, $\lambda \neq 1$, then $\lambda \in \sigma(yx + 1)$, so $\lambda - 1 \in \sigma(yx)$, hence $\lambda - 1 \in \sigma(xy)$. Thus $\sigma(xy)$ is unbounded, a contradiction.] Thus, for a set $(\mathcal{G}|\mathcal{R})$ where the relations contain constant terms, it must be verified that there is at least one representation in order for $C^*(\mathcal{G}|\mathcal{R})$ to be defined. (If the relations do not contain constant terms, there is always at least one representation, setting all generators equal to 0; if representations on the zero Hilbert space are allowed, the same is true about relations with constant terms.)

Stable Relations

VI.2.2.8. Semiprojectivity can be phrased in terms of stable relations. Let $(\mathcal{G}|\mathcal{R})$ be a set of generators and relations, which to avoid complications we will take to be finite. If $\mathcal{G} = \{x_1, \ldots, x_k\}$, then a set $\{y_1, \ldots, y_k\}$ in a C*-algebra A *approximately satisfies the relations within* $\delta > 0$ if, whenever

$$\|p(x_{i_1}, \cdots, x_{i_n}, x_{i_1}^*, \cdots, x_{i_n}^*)\| \leq \eta$$

is a relation in $\mathcal{R}$, we have

$$\|p(y_{i_1}, \cdots, y_{i_n}, y_{i_1}^*, \cdots, y_{i_n}^*)\| < \eta + \delta.$$

VI.2.2.9. Definition. The finite set $(\mathcal{G}|\mathcal{R})$ is *weakly stable* (or has *weakly stable relations*) if for any $\epsilon > 0$ there is a $\delta > 0$ such that, whenever A is a C*-algebra and $\{y_1, \ldots, y_k\} \subseteq A$ approximately satisfy the relations within δ, there are $x_1, \ldots, x_k \in A$ exactly satisfying the relations with $\|x_j - y_j\| < \epsilon$ for $1 \leq j \leq k$.

VI.2.2.10. It is easily seen that if $C^*(\mathcal{G}|\mathcal{R})$ is semiprojective, then $(\mathcal{G}|\mathcal{R})$ is weakly stable. The converse is not quite true.

The situation can be rephrased: for $\delta > 0$, let $\mathcal{R}_\delta$ be the set of relations in $\mathcal{R}$, "softened" by replacing each η by $\eta + \delta$. Then there is a natural homomorphism π_δ from $C^*(\mathcal{G}|\mathcal{R}_\delta)$ to $C^*(\mathcal{G}|\mathcal{R})$. The relations $\mathcal{R}$ are weakly stable if there is an approximate right inverse for π_δ for sufficiently small δ. It follows easily from the definition that if $C^*(\mathcal{G}|\mathcal{R})$ is semiprojective, then there is an exact right inverse for π_δ for small δ. If there is an exact right inverse for π_δ for some δ, the relations are said to be *stable*. It turns out that if $\mathcal{R}$ is finite and stable, then $C^*(\mathcal{G}|\mathcal{R})$ is semiprojective, so in the finitely presented case semiprojectivity is equivalent to stable relations.

See [Lor97] for a complete discussion of these and related matters.

VI.2.3. Projective C*-Algebras

The definition and study of projective C*-algebras did not begin with [Bla85b], and probably qualify as folklore. The following observations have long been well known.

VI.2.3.1. For any C*-algebra A, we define the *cone over* A to be the C*-algebra CA of continuous functions f from $[0, 1]$ to A with $f(0) = 0$. There is a surjective homomorphism from CA to A given by evaluation at 1, i.e. $A \cong CA/SA$, where $SA \cong C_0(\mathbb{R}, A)$ is the ideal of functions in CA vanishing at 1. The cone CA is contractible: set $h(s,t)(f) = f(st)$.

If A is projective (in $\mathcal{S}$), then there is a lift of the isomorphism $A \to CA/SA$ to $\psi : A \to CA$. Composing ψ with h, we get a contraction of A. Thus a projective C*-algebra is contractible, and in particular nonunital. (However, not every contractible C*-algebra is projective or even semiprojective: for example, the cone over $C([0, 1])$ is not semiprojective (); see also VI.2.3.5..) Thus projective C*-algebras are quite rare.

VI.2.3.2. Examples. (i) $C_0((0, 1])$ is the universal C*-algebra generated by a positive element of norm 1. It is well known that if B is a C*-algebra, J a closed ideal, and y a positive element in B/J of norm 1, there is a positive element x of B of norm 1 with $\pi(x) = y$, where $\pi : B \to B/J$ is the quotient map. [Let $z \in B$ with $\pi(z) = y^{1/2}$; then $\pi(z^*z) = y$, and we may take $x = f(z^*z)$, where $f(t) = t$ for $t \leq 1$ and $f(t) = 1$ for $t > 1$.] Thus $C_0((0, 1])$ is projective.

(ii) $C_0((0, 1]) \oplus C_0((0, 1])$ is the universal C*-algebra generated by a self-adjoint element of norm 1 (the two summands are generated by the positive and negative parts). It is again well known that if B is a C*-algebra, J a closed ideal, and y a self-adjoint element in B/J of norm 1, there is a self-adjoint element x of B of norm 1 with $\pi(x) = y$ [the argument is almost identical to the positive case]. Thus $C_0((0, 1]) \oplus C_0((0, 1])$ is projective.

VI.2.3.3. Proposition. If A_1 and A_2 are projective (and separable), then $A_1 \oplus A_2$ is projective. By iteration, any finite direct sum of projective C*-algebras is projective.

PROOF: If B is a C*-algebra, J an ideal in B, and $\phi : A_1 \oplus A_2 \to B/J$ a homomorphism, let y_1 and y_2 be strictly positive elements of A_1 and A_2 respectively, of norm 1. Then $\phi(y_1)$ and $\phi(y_2)$ are orthogonal positive elements of B/J of norm 1, and thus lift to orthogonal positive elements x_1 and x_2 of B of norm 1 by VI.2.3.2.(ii). If B_1 and B_2 are the hereditary C*-subalgebras of B generated by x_1 and x_2 respectively, and D_1 and D_2 the hereditary C*-subalgebras of B/J generated by $\phi(y_1)$ and $\phi(y_2)$ respectively, then $\pi : B \to B/J$ sends B_1 onto D_1 and B_2 onto D_2 [Ped79, 1.5.11]. If $\phi_1 = \phi|_{A_1 \oplus 0}$ and $\phi_2 = \phi|_{0 \oplus A_2}$, then ϕ_j maps A_j into D_j $(j = 1, 2)$, and lifts to $\psi_j : A_j \to B_j$, giving a lift of ϕ from $A_1 \oplus A_2$ to B. 👍

VI.2.3.4. PROPOSITION. [Lor97,] A (separable) projective C*-algebra is residually finite-dimensional ().

PROOF: Let l^2_n be the subspace of l^2 spanned by the first n standard basis vectors (i.e. $l^2_n = l^2(\{1, \ldots, n\})$, and let P_n be the projection of l^2 onto l^2_n. Then $P_n \to I$ in the strong-* topology, so for every $T \in \mathcal{B}(l^2)$ we have $P_n T P_n \to T$ in the strong-* topology. Regard $\mathcal{B}(l^2_n)$ as $P_n \mathcal{B}(l^2) P_n \subseteq \mathcal{B}(l^2)$. Let B be the set of all (bounded) sequences (T_n) in $\prod \mathcal{B}(l^2_n)$ such that (T_n) converges strong-* in $\mathcal{B}(l^2)$, and J the set of sequences in B which converge strong-* to 0. It is easily checked, using that multiplication is jointly strong-* continuous on bounded sets, that B is a C*-subalgebra of $\prod \mathcal{B}(l^2_n) \cong \prod \mathbb{M}_n$, J is a closed ideal of B, and $B/J \cong \mathcal{B}(l^2)$ (if $T \in \mathcal{B}(l^2)$, then $(P_n T P_n) \in B$).

If A is a separable C*-algebra, there is an injective map $\phi : A \to \mathcal{B}(l^2) \cong B/J$. If A is projective, ϕ lifts to an injective map $\psi : A \to \prod \mathbb{M}_n$, and the coordinate projections give a separating family of finite-dimensional representations of A. 👍

VI.2.3.5. COROLLARY. If A is a separable C*-algebra and CA is projective, then A is RFD.

VI.2.3.6. Projectivity in $\mathcal{S}_1$ is somewhat different. If A is a unital C*-algebra, define the *unital cone* $C^\dagger A$ of A to be the set of continuous functions from $[0, 1]$ to A such that $f(0)$ is a scalar multiple of 1 ($C^\dagger A$ is just the unitization $(CA)^\dagger$). As in the nonunital case, there is a contraction of $C^\dagger A$ to $\mathbb{C}$, i.e. $C^\dagger A$ is homotopy equivalent to $\mathbb{C}$. If J is the ideal of $C^\dagger A$ of functions f with $f(1) = 0$, then $[C^\dagger A]/J \cong A$ by evaluation at 1. If A is projective in $\mathcal{S}_1$, there is a unital lift of the identification, which gives a contraction of A to $\mathbb{C}$, i.e. A is homotopy equivalent to $\mathbb{C}$.

If A is projective (in $\mathcal{S}$), it is an easy argument that $A^\dagger$ is projective in $\mathcal{S}_1$, cf. VI.2.4.5.. The converse is also true if A is nonunital (the proof is a simplified version of the proof of VI.2.4.5.). Thus $C([0, 1]) = C_0((0, 1])^\dagger$ is projective in $\mathcal{S}_1$.

VI.2.4. Low-Hanging Fruit

One nice thing about opening up a new area of mathematical research is that there is usually some low-hanging fruit to be grabbed. Some of this was done in [Bla85b].

VI.2.4.1. The first observations concern the simplest C*-algebra, $\mathbb{C}$. It is trivial that $\mathbb{C}$ is projective in the category $\mathcal{S}_1$. However, the case of the category $\mathcal{S}$ is a little more complicated. Note first that a *-homomorphism ϕ from $\mathbb{C}$ to a C*-algebra B is effectively the same thing as a choice of a projection ($\phi(1)$) in B. So projectivity of $\mathbb{C}$ in $\mathcal{S}$ is exactly the question of whether projections lift from quotients. But, as is well known, they do not in general: probably the simplest example is $B = C([0,1])$, $J = C_0((0,1))$. The projection $(1,0)$ in $B/J \cong \mathbb{C}^2$ does not lift to a projection in B. Thus $\mathbb{C}$ is not projective (in the category $\mathcal{S}$). However, we have the following fact (which may qualify as low-hanging fruit but is not lying on the ground):

VI.2.4.2. Proposition. Let B, (J_n) , J form a tower (VI.2.1.1.). Write $\pi_n : B \to B/J_n$ for each n, and π the quotient map from B (or any B/J_n) to B/J. If q is a projection in B/J, then for some sufficiently large n there is a projection p in B/J_n with $\pi(p) = q$.

Proof: For any $x \in B$, the sequence $(\|\pi_n(x)\|)$ is decreasing, and it follows immediately from the definition of the quotient norm that

$$\|\pi(x)\| = \inf_n \|\pi_n(x)\| = \lim_{n\to\infty} \|\pi_n(x)\| \ .$$

Let $x = x^* \in B$ with $\pi(x) = q$ (we may even choose $x \geq 0$ and $\|x\| = 1$ if desired, but this is not necessary). Then $\|\pi(x - x^2)\| = \|\pi(x) - \pi(x)^2\| = \|q - q^2\| = 0$, so for some sufficiently large n we have

$$\|\pi_n(x - x^2)\| = \|\pi_n(x) - \pi_n(x)^2\| < \frac{1}{4}$$

so the self-adjoint element $\pi_n(x)$ in B/J_n has a gap in its spectrum around $\frac{1}{2}$. If $f : \mathbb{R} \to \mathbb{R}$ is the function with $f(t) = 0$ for $t < \frac{1}{2}$ and $f(t) = 1$ for $t > \frac{1}{2}$, then f is continuous on $\sigma(\pi_n(x))$, so we make the element $p = f(\pi_n(x))$ in B/J_n by functional calculus. Then p is a projection, and by II.1.6.2.(iv)

$$\pi(p) = \pi(f(\pi_n(x)) = f(\pi(x)) = f(q) = q \ .$$

👍

VI.2.4.3. Corollary. $\mathbb{C}$ is semiprojective.

VI.2.4.4. Corollary. Let A be a (separable) unital C*-algebra. Then A is semiprojective (in $\mathcal{S}$) if and only if it is semiprojective in $\mathcal{S}_1$.

Proof: Suppose A is semiprojective in $\mathcal{S}_1$. Let B, (J_n), J, and ϕ be as in the definition VI.2.1.1. (for $\mathcal{S}$). Set $q = \phi(1_A)$, and fix m and projection $p \in B/J_m$ with $\pi(p) = q$. Let $D = p(B/J_m)p$ and for each n let $K_n = p(J_{n+m}/J_m)p$. Then D is unital, (K_n) is an increasing sequence of closed ideals in D with closed union $K = p(J/J_m)p$. D/K_n is naturally isomorphic to a corner in B/J_{m+n}, and $D/K \cong q(B/J)q$. The range of ϕ is contained in $q(B/J)q$, so ϕ may be regarded as a unital *-homomorphism into D/K, which partially lifts to a unital *-homomorphism $\psi : A \to D/K_n$ by assumption. Then ψ can be regarded as a *-homomorphism from A to a corner in B/J_{m+n} which gives a partial lift of ϕ.

The converse seems trivial, but there is a subtlety. Suppose A is unital and semiprojective, and B, (J_n), J, and ϕ are as in the definition of semiprojectivity, with B and ϕ unital. Semiprojectivity gives a partial lift ψ of ϕ to a *-homomorphism of A to B/J_n for some n. But we need a *unital* lift. However, if $\psi(1_A) = p$, then $\pi(p) = 1 = \pi(1)$, so $\pi(1-p) = 0$. Thus, by the argument in the proof of VI.2.4.2., $\|\pi_m(1-p)\| < 1$ for some sufficiently large $m > n$. But $\pi_m(1-p)$ is a projection, so $\pi_m(1-p) = 0$, $\pi_m(p) = 1$, and $\pi_m \circ \psi$ is a unital lift of ϕ to B/J_m. 👍

Thus, to prove that a unital C*-algebra A is semiprojective, it suffices to check that the definition is satisfied when B and ϕ are unital.

In fact, we have:

VI.2.4.5. Corollary. Let A be a (separable) C*-algebra. Then A is semiprojective if and only if $A^\dagger$ is semiprojective.

Proof: Suppose A is semiprojective. To show that $A^\dagger$ is semiprojective, it suffices to show it is semiprojective in $\mathcal{S}_1$. If B, (J_n), J form a unital tower and $\phi : A^\dagger \to B/J$ is a unital *-homomorphism, then $\phi|_A$ partially lifts to a *-homomorphism ψ from A to B/J_n for some n. Then ψ extends to a unital *-homomorphism from $A^\dagger$ to B/J_n, which gives a partial lift of ϕ.

Conversely, suppose $A^\dagger$ is semiprojective. We may assume A is nonunital. Let B, (J_n), J form a tower, and let $\phi : A \to B/J$ be a *-homomorphism, which we may take to be an isomorphism (VI.2.1.8.). Then $B^\dagger$, (J_n), J form a unital tower, and ϕ extends to a unital homomorphism $\tilde{\phi}$ from $A^\dagger$ to $B^\dagger/J \cong (B/J)^\dagger$. Partially lift $\tilde{\phi}$ to $\tilde{\psi} : A^\dagger \to B^\dagger/J_n$ for some n, and let $\psi = \tilde{\psi}|_A$. To show that $\psi(A)$ is contained in B/J_n, note that the diagram

$$\begin{tikzcd}
(B/J_n)^\dagger \arrow[r, "\rho_n"] \arrow[d] & (B/J_n)^\dagger/(B/J_n) \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "id"] \\
(B/J)^\dagger \arrow[r, "\rho"] & (B/J)^\dagger/(B/J) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

is commutative, where the ρ's are the quotient maps and the last horizontal arrows are the canonical isomorphisms. Thus the diagram

$$\begin{tikzcd}
 & (B/J_n)^\dagger \arrow[r, "\rho_n"] \arrow[d] & (B/J_n)^\dagger/(B/J_n) \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "id"] \\
A \arrow[r, "\phi"] \arrow[ur, dotted, "\psi"] & (B/J)^\dagger \arrow[r, "\rho"] & (B/J)^\dagger/(B/J) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

is commutative. Since $\rho \circ \phi$ is the zero map, it follows that $\rho_n \circ \psi$ is also the zero map. 👍

Note that the proof in [Bla85b] (and in [Lor97] and [ST12], which just cite the argument of [Bla85b]) is slightly incomplete.

The next piece of low-hanging fruit is to extend VI.2.4.2. to finite families of mutually orthogonal projections.

VI.2.4.6. Proposition. Let B, J_n, and J form a tower, and let $q_1, \ldots, q_k$ be mutually orthogonal projections in B/J. Then for sufficiently large n, there are mutually orthogonal projections $p_1, \ldots, p_k$ in B/J_n with $\pi(p_j) = q_j$ for all j. If B (and hence B/J) is unital and $q_1 + \cdots + q_k = 1$, then we may choose the p_j so that $p_1 + \cdots + p_k = 1$.

Proof: First lift q_1 to a projection r_1 in B/J_{n_1}. Cut down B/J_{n_1} to $(1 - r_1)(B/J_{n_1})(1 - r_1)$ (note that $1 - r_1$ is a projection in the unitization in general) and all succeeding algebras similarly, and B/J to $(1 - q_1)(B/J)(1 - q_1)$. We may now lift q_2 to a projection r_2 in $(1 - \pi_{n_2}(r_1))(B/J_{n_2})(1 - \pi_{n_2}(r_1))$ for some $n_2 > n_1$, i.e. to a projection r_2 in B/J_{n_2} orthogonal to $\pi_{n_2}(r_1)$. Continue inductively for k steps and set $p_j = \pi_{n_k}(r_j)$ for $1 \leq j \leq k$. In the last case stop after $k - 1$ steps and take $p_k = 1 - p_1 - \cdots - p_{k-1}$. 👍

Another related piece of low-hanging fruit says that partial isometries can be partially lifted to have as source and range projections specified partial lifts of the source and range projections in the quotient:

VI.2.4.7. Proposition. Let B, J_n, and J form a tower. Let q_0, q_1 be projections in B/J, and v a partial isometry in B/J with $v^*v = q_0$, $vv^* = q_1$. Let p_0, p_1 be projections in B/J_n with $\pi(p_j) = q_j$ $(j = 0, 1)$. Then there is an $m \geq n$ and a partial isometry $u \in B/J_m$ with $u^*u = \pi_m(p_0)$, $uu^* = \pi_m(p_1)$, $\pi(u) = v$.

As a special case if B is unital, any isometry partially lifts to an isometry, and any unitary partially lifts to a unitary.

Proof: Let $x \in B/J_n$ with $\pi(x) = v$. Then, for large $m > n$,

$$\|\pi_m(x)^*\pi_m(x) - \pi_m(p_0)\| \text{ and } \|\pi_m(x)\pi_m(x)^* - \pi_m(p_1)\|$$

are small ($< \frac{1}{8}$ will do). Thus $\sigma(\pi_m(x)^*\pi_m(x))$ is concentrated near 0 and 1. If f is the function on this spectrum which is 0 on the part near 0 and $f(t) = t^{-1}$ on the part near 1, then $w = \pi_m(x)f(\pi_m(x)^*\pi_m(x))$ is a partial isometry with $w^*w = p_0'$ and $ww^* = p_1'$ for projections p_0' and p_1' close to $\pi_m(p_0)$ and $\pi_m(p_1)$ respectively. If $z_j = y_j(y_j^*y_j)^{-1/2}$ $(j = 0, 1)$, where $y_j = (2p_j' - 1)(2\pi_m(p_j) - 1)$, then the z_j are unitaries in $(B/J_m)^\dagger$ conjugating $\pi_m(p_j)$ to p_j' (cf. ()), and $\pi(z_j) = 1$ in $(B/J)^\dagger$. Then $u = z_1^*wz_0$ is the desired lift in B/J_m. 👍

VI.2.4.8. By applying VI.2.4.2., VI.2.4.6., and VI.2.4.7. successively with a bit of care to the generators and relations in the standard presentations, it follows that the following C*-algebras are semiprojective:

Any finite-dimensional C*-algebra.

$C(\mathbb{T})$.

$C^*(\mathbb{F}_n)$ for *n finite*.

The Toeplitz algebra.

The Cuntz algebras O_n (*n finite*).

The Cuntz-Krieger algebras O_A for any A (not necessarily irreducible).

To illustrate, consider $\mathbb{M}_k$. Let $\{e_{ij}\}$ be the standard matrix units. If (B, J_n, J) is a tower, first partially lift $\{e_{11}, e_{22}, \dots, e_{kk}\}$ to mutually orthogonal projections $\{f_{11}, f_{22}, \dots, f_{kk}\}$ in B/J_n by VI.2.4.6.. Then, repeatedly increasing n if necessary, successively lift each e_{1j} $(2 \leq j \leq k)$ to a partial isometry f_{1j} in B/J_n with $f_{1j}^* f_{1j} = f_{jj}$ and $f_{1j} f_{1j}^* = f_{11}$ for $1 \leq j \leq k$. Then set $f_{ij} = f_{1i}^* f_{1j}$ for all i, j. The other examples are similar and the argument is left to the reader.

VI.2.4.9. Having a finite presentation is crucial in applying these results successively, since the n may have to be increased each time a new projection or partial isometry is partially lifted. In fact, $\mathbb{K}$ and $C^*(\mathbb{F}_\infty)$, the universal C*-algebra generated by a sequence of unitaries, are not semiprojective. This fact, while quite believable, is nontrivial to prove (VI.2.4.22.). An indication that the situation is delicate is that by the same argument O_∞, the universal C*-algebra generated by a sequence of isometries with mutually orthogonal range projections, should also fail to be semiprojective; but it is in fact semiprojective ()! (A key difference is that the K-theory of O_∞ is finitely generated but the K-theory of $C^*(\mathbb{F}_\infty)$ is not. A different argument is needed for $\mathbb{K}$ (VI.2.4.19.).)

VI.2.4.10. If D is a finite-dimensional C*-algebra and $\{æ_{ij}^{(r)}\}$ is a set of elements of a C*-algebra A, indexed like a set of matrix units of type D (VI.2.2.5.(iv)), then $\{æ_{ij}^{(r)}\}$ is a set of *approximate matrix units of type D within δ* if the $æ_{ij}^{(r)}$ approximately satisfy the matrix unit relations within δ. Semiprojectivity of D implies:

VI.2.4.11. Proposition. Let D be a finite-dimensional C*-algebra. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that, whenever A is a C*-algebra and $\{æ_{ij}^{(r)}\}$ is a set of approximate matrix units of type D within δ in A, then there is a set $\{e_{ij}^{(r)}\}$ of exact matrix units of type D in A with $\|e_{ij}^{(r)} - æ_{ij}^{(r)}\| < \epsilon$ for all i, j, r.

This result is used, for example, in the theory of AF algebras. Recall (**??**(iv)) that an AF algebra is a C*-algebra which is isomorphic to an inductive limit of a sequence of finite-dimensional C*-algebras. Here is a "local" description:

VI.2.4.12. Definition. A C*-algebra A is an *AF algebra in the local sense* (*local AF algebra*) if, for every $x_1, \dots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-subalgebra B and elements $y_1, \dots, y_n \in B$ with $\|x_i - y_i\| < \epsilon$ for $1 \leq i \leq n$.

Permanence

VI.2.4.13. There is an exact noncommutative version of the special case of VI.1.2.3. that a retract of an ANR is an ANR. If X is a space and Y a closed subspace, there is an exact sequence

$$0 \longrightarrow C_0(X \setminus Y) \longrightarrow C(X) \xrightarrow{\sigma} C(Y) \longrightarrow 0$$

where σ is restriction of functions and $C_0(X \setminus Y)$ is identified with the elements of $C(X)$ which vanish on Y. If there is a retraction $r : X \to Y$, the homomorphism $\rho : C(Y) \to C(X)$ defined by $\rho(f) = f \circ r$ is a cross section for σ, i.e. the exact sequence splits. The converse also holds: retractions exactly correspond to split exact sequences. In the noncommutative case, we may thus regard a split exact sequence

$$0 \longrightarrow J \longrightarrow A \underset{\sigma}{\overset{\rho}{\rightleftarrows}} D \longrightarrow 0$$

of C*-algebras as giving a *retraction* of A to D, and say that D is a *retract* of A. This is precisely the case where there are a (necessarily surjective) *-homomorphism $\sigma : A \to D$ and a (necessarily injective) *-homomorphism $\rho : D \to A$ such that $\sigma \circ \rho = id_D$.

VI.2.4.14. Proposition. Let D be a retract of a C*-algebra A. If A is semiprojective, so is D.

Proof: Let $\sigma : A \to D$ and $\rho : D \to A$ give the retraction. Let B, (J_n), J be as in VI.2.1.1., and $\phi : D \to B/J$ a *-homomorphism. Then $\phi \circ \sigma : A \to B/J$ partially lifts to a *-homomorphism $\theta : A \to B/J_n$ for some n. The *-homomorphism $\psi = \theta \circ \rho$ is a partial lift of ϕ.

See [Bla85b, 2.37] for a more general result in the unital case which is an exact analog of VI.1.2.3.. The proof (and even the statement) uses multiplier algebras, and is not a direct translation of the proof of VI.1.2.3.. This result is interesting since an ideal in a semiprojective C*-algebra is not necessarily semiprojective (e.g. $\mathbb{K}$ is an ideal in the Toeplitz algebra $\mathcal{T}$; there are commutative examples too).

VI.2.4.15. Proposition. Let A_1 and A_2 be (separable) C*-algebras. Then $A_1 \oplus A_2$ is semiprojective if and only if A_1 and A_2 are semiprojective. By iteration, a finite direct sum of semiprojective C*-algebras is semiprojective.

Proof: A_1 and A_2 are retracts of $A_1 \oplus A_2$, so if $A_1 \oplus A_2$ is semiprojective, so are A_1 and A_2. The proof of the converse is almost identical to the proof of VI.2.3.3..

VI.2.4.16. Proposition. Let A be a unital semiprojective C*-algebra. Then $M_k(A)$ is semiprojective for every k.

Proof: Fix k, and let $\{e_{ij}\}$ be the matrix units of $\mathbb{M}_k \subseteq M_k(A)$. Let B, (J_n), J be as in VI.2.1.1., and $\phi : M_k(A) \to B/J$ a *-homomorphism. First lift $\{\phi(e_{ij})\}$ to a set $\{f_{ij}\}$ of matrix units in B/J_n for some n. Set $D = f_{11}(B/J_n)f_{11}$, $K_m = f_{11}(J_m/J_n)f_{11}$ for $m > n$, $K = f_{11}(J/J_n)f_{11} = [\cup_n K_n]^-$. Then ϕ defines a unital *-homomorphism ϕ_1 from $e_{11}M_k(A)e_{11} \cong A$ to D/J, so it lifts to a unital *-homomorphism ψ_1 from $e_{11}M_k(A)e_{11}$ to $D/J_m \cong \pi_m(f_{11})(B/J_m)\pi_m(f_{11})$ for some $m > n$. Every element of $M_k(A)$ can be uniquely written as $\sum_{i,j} e_{i1}a_{ij}e_{1j}$ for elements $a_{ij} \in e_{11}M_k(A)e_{11}$. Set

$$\psi\left(\sum_{i,j} e_{i1}a_{ij}e_{1j}\right) = \sum_{i,j} \pi_m(f_{i1})\psi_1(a_{ij})\pi_m(f_{1j}) \in B/J_m \ .$$

The unital hypotheses in VI.2.4.16. can be eliminated with considerable work ().

Maps into Inductive Limits

The next fact, which has proved extremely useful, qualifies as low-hanging fruit because its proof was already done in the commutative case, although the commutative version of this argument is not easy to find in the literature (the usual commutative proof, e.g. in [MS82], is a discrete version combined with ()) and I frankly cannot remember where or even if I saw it while writing [Bla85b], so I cannot properly attribute it. At least the crucial technical tool in the argument, the mapping telescope, had already appeared, due to L. Brown (cf. [Sch84, 5.2]). Both the commutative and noncommutative versions are key technical tools in the development of shape theory.

VI.2.4.17. Theorem. [Bla85b, 3.1] Let A be a semiprojective C*-algebra, and $(B_n, \beta_{m,n})$ be an inductive system of C*-algebras. Let

$$B = \varinjlim (B_n, \beta_{m,n}).$$

If $\phi : A \to B$ is a homomorphism, then for all sufficiently large n there are homomorphisms $\phi_n : A \to B_n$ such that $\beta_n \circ \phi_n$ is homotopic to ϕ and converges pointwise to ϕ as $n \to \infty$, where β_n is the standard map from B_n to B.

Proof: Let D be the C*-subalgebra of $\prod_n C([n, n+1], B_n)$ consisting of all sequences (f_n) such that $f_{n+1}(n+1) = \beta_{n,n+1}(f_n(n+1))$ for all n and such that

$$\lim_{s,t\to\infty,\, s\le t} \|f_n(t) - \beta_{m,n}(f_m(s))\| = 0$$

where $m \le s \le m+1$, $n \le t \le n+1$, $m \le n$. (If the connecting maps are injective, so each B_n can be considered a C*-algebra of B, D can be more simply described as

$$\{f \in C([0, +\infty], B) : \forall n \,\forall t \le n,\ f(t) \in B_n\}\ .$$

D is called a *mapping telescope*.) For $k \in \mathbb{N}$ set

$$J_k = \{(f_n) \in D : f_n \equiv 0 \text{ for } n \ge k\}$$

$$J = \{(f_n) \in D : \lim_{t\to\infty} \|f_n(t)\| = 0\} = [\cup_k J_k]^-\ .$$

We have $D/J \cong B$, and ϕ defines a *-homomorphism from A to D/J, which lifts to D/J_k for some k, which is isomorphic to the C*-subalgebra of $\prod_{n\ge k} C([n, n+1], B)$ defined by the same conditions as in D. Composing with the evaluation map at n gives the map ϕ_n for all $n \ge k$, and these maps clearly have the desired properties: the homotopy is given by evaluation at t $(t \ge k)$ composed with β_r $(r \le t \le r+1)$. 👍

VI.2.4.18. The connecting maps in the inductive system of VI.2.4.17. are not required to be injective. In fact, if B, (J_n), J form a tower, the B/J_n form an inductive system

$$B/J_1 \longrightarrow B/J_2 \longrightarrow \cdots$$

with *surjective* connecting maps, and the inductive limit is B/J. In this case a stronger conclusion holds (by definition). In the general case where the union of the images of the B_n does not contain $\phi(A)$, no stronger conclusion is possible in general.

VI.2.4.19. A simple application shows that $\mathbb{K}$ is not semiprojective. Write $\mathbb{K} = \lim_{\to}(\mathbb{M}_n, \alpha_n)$ as in (). If $\mathbb{K}$ were semiprojective, the identity map on $\mathbb{K}$ would approximately factor through $\mathbb{M}_n$. But there is no nonzero homomorphism from $\mathbb{K}$ to $\mathbb{M}_n$ for any n. An identical argument shows that no infinite-dimensional AF algebra is semiprojective. By a slight extension, no infinite-dimensional simple ASH algebra can be semiprojective (if it were, it would have a finite-dimensional quotient). Since every stably finite classifiable simple C*-algebra is ASH, no stably finite classifiable simple C*-algebra is semiprojective. The purely infinite ones are a different story (VI.2.4.8.): see () for the full answer.

The result VI.2.4.17. holds also for generalized inductive limits in the sense of (), at least the pointwise convergence part (since the connecting maps are not homomorphisms in this case, it is unclear how the homotopy part could be phrased). Thus no infinite-dimensional simple MF algebra can be semiprojective.

Finite Generation of K-Theory

We can get an interesting consequence of VI.2.4.17. which is relevant to classification. We say a C*-algebra A has *finitely generated K-theory* if both $K_0(A)$ and $K_1(A)$ are finitely generated. While it does not appear to be known whether every semiprojective C*-algebra has finitely generated K-theory (it is true for ANRs by ()), we have:

VI.2.4.20. Corollary. Let A be a semiprojective C*-algebra. If A is an inductive limit of C*-algebras with finitely generated K-theory, then A itself has finitely generated K-theory.

Proof: If $A = \lim_{\to} A_n$, an application of VI.2.4.17. shows that $K_i(A)$ $(i = 0, 1)$ is isomorphic to a subgroup of $K_i(A_n)$ for some n. But a subgroup of a finitely generated abelian group is finitely generated.

Combining this with the classification and the construction in (), we obtain:

VI.2.4.21. Corollary. Let A be a semiprojective UCT Kirchberg algebra. Then A has finitely generated K-theory.

The converse also holds ().

VI.2.4.22. We also get a proof from this that $C^*(\mathbb{F}_\infty)$ is not semiprojective. We have $C^*(\mathbb{F}_\infty) = \lim_{\to} C^*(\mathbb{F}_n)$ in the obvious way. By (), $C^*(\mathbb{F}_n)$ has finitely generated K-theory, but $C^*(\mathbb{F}_\infty)$ does not.

Homotopy of Close Homomorphisms

We can also obtain a noncommutative analog of () by the same methods. A weak version (essentially Lemma VI.2.4.25.) was in [Bla85b] and the full result in [**?**]; the proof here is more elementary than the one there.

VI.2.4.23. Theorem. Let A be a semiprojective C*-algebra generated by a finite or countable set $\mathcal{G} = \{x_1, x_2, \dots\}$ with $\lim_{j\to\infty} \|x_j\| = 0$ if $\mathcal{G}$ is infinite. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that, whenever B is a C*-algebra, ϕ_0 and ϕ_1 *-homomorphisms from A to B with $\|\phi_0(x_j) - \phi_1(x_j)\| < \delta$ for all j, then there is a point-norm continuous path (ϕ_t) $(0 \le t \le 1)$ of *-homomorphisms from A to B connecting ϕ_0 and ϕ_1 with $\|\phi_t(x_j) - \phi_0(x_j)\| < \epsilon$ for all j for any $t \in [0, 1]$. (The δ depends on ϵ, A, and the set $\mathcal{G}$ of generators, but not on the B, ϕ_0, ϕ_1.)

VI.2.4.24. Corollary. Let A be a semiprojective C*-algebra generated by a finite or countable set $\mathcal{G} = \{x_1, x_2, \dots\}$ with $\lim_{j\to\infty} \|x_j\| = 0$ if $\mathcal{G}$ is infinite. Then there is a $\delta > 0$ such that, whenever B is a C*-algebra, ϕ_0 and ϕ_1 *-homomorphisms from A to B with $\|\phi_0(x_j) - \phi_1(x_j)\| < \delta$ for all j, then ϕ_0 and ϕ_1 are homotopic. (The δ depends on A and the set $\mathcal{G}$ of generators, but not on the B, ϕ_0, ϕ_1.)

Proof: Fix any $\epsilon > 0$, say $\epsilon = 1$, and apply VI.2.4.23.. 👍

In the proofs of the commutative versions of these results, a metric is fixed on the space and the δ depends on ϵ and the choice of metric. Fixing a set of generators whose norms go to 0 can be regarded as an analog of fixing a metric in our setting. Note that if the norms of the generators go to 0, the condition that $\|\phi_0(x_j) - \phi_1(x_j)\| < \delta$ for all j is really a finite condition since it is automatic for all j with $\|x_j\| < \frac{\delta}{2}$.

VI.2.4.25. Lemma. Let A be a semiprojective C*-algebra, B any C*-algebra, and (ϕ_n) a sequence of *-homomorphisms from A to B converging to a *-homomorphism ϕ in the point-norm topology. Then there is an m and a point-norm continuous path (ϕ_t) $(t \geq m)$ of homomorphisms agreeing with the ϕ_n for $t \in \mathbb{N}$ with $\phi_t \to \phi$ (point-norm) as $t \to \infty$. In particular, ϕ_n is homotopic to ϕ for all sufficiently large n.

Proof: Let $D = C([0,\infty], B)$,

$$J_n = \{f \in D : f(k) = 0\ \forall k, f \equiv 0 \text{ on } [n,\infty]\}$$

$$J = \{f \in D : f(k) = 0\ \forall k\} = [\cup_n J_n]^- \ .$$

Then D/J is isomorphic to the C*-algebra of convergent sequences in B. The ϕ_n naturally define a *-homomorphism from A to D/J. Lift to $\psi : A \to D/J_m$ for some m. 👍

We now give the proof of VI.2.4.23..

Proof: Suppose the conclusion is false for some $\epsilon > 0$. Then for each n there is a C*-algebra B_n and homomorphisms $\phi_{n,0}, \phi_{n,1}$ from A to B_n with $\|\phi_{n,0}(x_j) - \phi_{n,1}(x_j)\| < \frac{1}{n}$ for all j but that $\phi_{n,0}$ and $\phi_{n,1}$ are not homotopic via a homotopy $(\phi_{n,t})$ with $\|\phi_{n,t}(x_j) - \phi_{n,0}(x_j)\| < \epsilon$ for all t and j. Set $B = \prod B_n$. Define $\alpha_n, \alpha : A \to B$ by

$$\alpha_n(x) = (\phi_{1,0}(x), \dots, \phi_{n,0}(x), \phi_{n+1,1}(x), \dots, \phi_{m,1}, \dots)$$

$$\alpha(x) = (\phi_{1,0}(x), \dots, \phi_{n,0}(x), \phi_{n+1,0}(x), \dots, \phi_{m.0}, \dots)$$

for $x \in A$. Then $\alpha_n(x_j) \to \alpha(x_j)$ for all j, and hence $\alpha_n \to \alpha$ pointwise. But then the α_n extend to a path (α_t) converging point-norm to α as $t \to \infty$ by VI.2.4.25.. Let $\{x_1, \dots, x_r\}$ be the elements of $\mathcal{G}$ of norm $\geq \frac{\epsilon}{2}$. Then there is an m such that $\|\alpha_t(x_j) - \alpha(x_j)\| < \epsilon$ for all $t \geq m$ and $1 \leq j \leq r$. We automatically have $\|\alpha_t(x_j) - \alpha(x_j)\| < \epsilon$ for any t and any j with $\|x_j\| < \frac{\epsilon}{2}$, i.e. the inequality holds for all $x_j \in \mathcal{G}$ for $t \geq m$. We thus have a contradiction when the α_t are projected to B_m. 👍

VI.2.5. Bad Apples

Some natural relations are not stable. The most notorious and relevant example for our purposes is the commutation relation.

The Voiculescu Matrices

VI.2.5.1. In [**?**], D. Voiculescu considered the following two matrices in $\mathbb{M}_n$. Let u_n be the cyclic shift matrix

$$u_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and let $v_n = diag(\omega_n, \omega_n^2, \dots, \omega_n^n)$, where $\omega_n = e^{\frac{2\pi i}{n}}$, a primitive n'th root of unity in $\mathbb{C}$. Then u_n and v_n are unitaries, and it is easily calculated that

$$\|u_n v_n - v_n u_n\| = \|v_n - u_n^* v_n u_n\| = |\omega_n - 1| < \frac{2\pi}{n}$$

so u_n and v_n are *almost commuting* unitaries. But it was shown in [**?**] that if $\tilde{u}_n$ and $\tilde{v}_n$ are exactly commuting unitaries, then

$$\max(\|u_n - \tilde{u}_n\|, \|v_n - \tilde{v}_n\|) \geq 1$$

i.e. u_n and v_n are not *nearly commuting* unitaries. This example was discussed and extended in [EL91], where a K-theoretic obstruction to being nearly commuting was identified.

VI.2.5.2. We can thus do the following construction. Let

$$B = \prod_n \mathbb{M}_n = \{(x_1, x_2, \dots) : x_n \in \mathbb{M}_n,\ \sup_n \|x_n\| < \infty\}$$

$$J_n = \{(x_1, x_2, \dots) : x_n \in \mathbb{M}_n,\ x_k = 0 \text{ for } k > n\}$$

$$J = \bigoplus_n \mathbb{M}_n = \{(x_1, x_2, \dots) : x_n \in \mathbb{M}_n,\ \lim_{n\to\infty} \|x_n\| = 0\}$$

Then B, (J_n), J form a tower. Set $u = \pi((u_1, u_2, \dots))$, $v = \pi((v_1, v_2, \dots))$, where $\pi : B \to B/J$ is the quotient map and u_n, v_n are the Voiculescu matrices. Then u and v are commuting unitaries in B/J, and thus define a *-homomorphism ϕ from $C(\mathbb{T}^2)$, the universal C*-algebra generated by two commuting unitaries, to B/J. But ϕ cannot partially lift to $B/J_n \cong \prod_{k>n} \mathbb{M}_k$ since a lift would give sequences $(\tilde{u}_k)$ and $(\tilde{v}_k)$ of commuting unitaries in $\mathbb{M}_k$ for $k > n$ with $\|u_k - \tilde{u}_k\| \to 0$ and $\|v_k - \tilde{v}_k\| \to 0$, contradicting Voiculescu's result. Thus $C(\mathbb{T}^2)$ is not semiprojective (in $\mathcal{S}$), even though $\mathbb{T}^2$ is an ANR. In fact, the result that almost commuting unitaries are not necessarily nearly commuting means that the standard presentation

$$(\{u, v, 1\} | \{u1 = 1u = v1 = 1v = 11 = 1, u^*u = uu^* = v^*v = vv^* = 1, uv = vu\})$$

of $C^*(\mathbb{T}^2)$ is not weakly stable in the sense of VI.2.2.9..

VI.2.5.3. For another example (from [Bla85b]), let s be the unilateral shift, so $C^*(s)$ is the Toeplitz algebra $\mathcal{T}$ (). $C^*(s)$ contains $\mathbb{K}$ as an ideal, and the quotient is isomorphic to $C(\mathbb{T})$. Let B be the C*-algebra of all sequences in $C^*(s)$ converging to a scalar multiple of the unit,

$$J_n = \{(x_1, x_2, \dots) : x_k \in \mathbb{K} \subseteq C^*(s)\ \forall k, x_k = 0\ \forall k > n\}$$

$$J = \{(x_1, x_2, \dots) : x_k \in \mathbb{K} \subseteq C^*(s)\ \forall k, x_k \to 0\} \ .$$

Then B, (J_n), J form a tower. Let $x_n = Re(s)/n$ and $y_n = Im(s)/n$. Then x_n and y_n are self-adjoint contractions, $x_n \to 0$, $y_n \to 0$, and x_n and y_n commute mod $\mathbb{K}$ since s is a unitary and thus normal mod $\mathbb{K}$. Set $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ in B; then x and y are self-adjoint contractions which commute mod J, so there is a unital *-homomorphism ϕ from $C([-1,1]^2)$ to B/J sending the coordinate function generators to x and y respectively. But x and y are not the images of commuting self-adjoint contractions in B/J_n for any n since no compact perturbation of s is normal. Thus $C([-1,1]^2) \cong C([0,1]^2)$ is not semiprojective.

In fact, the following result, predicted in [Bla85b], was proved by A. Sørensen and H. Thiel [ST12]:

VI.2.5.4. Theorem. Let X be a (compact metrizable) space. Then $C(X)$ is semiprojective (in $\mathcal{S}$) if and only if X is an ANR of dimension ≤ 1.

We will not give the proof since a more general result will be obtained (). But the rough idea is that $C(X)$ has a presentation which does not include commutation relations if and only if $dim(X) \leq 1$. (If $C(X)$ is semiprojective, then X must be an ANR by VI.2.1.5..)

VI.2.5.5. Instability of commutation relations is actually a rather delicate matter, and depends on presence or absence of other relations. For example, if A and B are unital C*-algebras, the natural way to get a presentation of $A \otimes B$ ($A \otimes_{\max} B$ in the nonnuclear case) is to take presentations of A and B and add commutation relations between the generators of A and those of B (using that $A \otimes_{\max} B$ is the universal C*-algebra generated by commuting unital copies of A and B). Thus it should be difficult for $A \otimes B$ to be semiprojective even if A and B are unital and semiprojective, e.g. $C([0,1]) \otimes C([0,1]) \cong C([0,1]^2)$. But $A \otimes B$ can be semiprojective, e.g. if A is semiprojective and B is finite-dimensional (hence also semiprojective). For a deeper example, $O_n \otimes O_2 \cong O_2$ is semiprojective for any n. Actually $O_n \otimes O_m$ is semiprojective for any n and m; in fact, the tensor product of any two semiprojective UCT Kirchberg algebras is semiprojective, as follows easily from () and the Künneth Theorem for Tensor Products (). Incidentally, there are UCT Kirchberg algebras A and B, neither of which is semiprojective, for which $A \otimes B \cong O_2$ and hence is semiprojective. See also () (stable Kirchberg)

More Bad Apples

Sometimes commutation relations can be hidden. Consider the following interesting example:

VI.2.5.6. Example. Consider the universal unital C*-algebra A generated by four positive elements a_1, a_2, b_1, b_2 of norm 1, such that $a_1 \perp a_2$, $b_1 \perp b_2$, $a_1 + a_2 + b_1 + b_2 = 1$. Formally, we may take as generators $\{a_1, a_2, b_1, b_2, 1\}$ with the following relations:

$$1 = 1^* = 1^2 \ , \quad 1a_j = a_j 1 = a_j \ , \quad 1b_j = b_j 1 = b_j \quad (j = 1, 2)$$

$$a_j = a_j^* \ , \quad \|a_j\| \leq 1 \ , \quad \|1 - a_j\| \leq 1 \quad (j = 1, 2)$$

$$b_j = b_j^* \ , \quad \|b_j\| \leq 1 \ , \quad \|1 - b_j\| \leq 1 \quad (j = 1, 2)$$

$$a_1 a_2 = b_1 b_2 = 0 \ (j = 1, 2) \ , \quad a_1 + a_2 + b_1 + b_2 = 1 \ .$$

All these relations are individually partially liftable (only the last takes work beyond what we have already done: if $\tilde{a}_j, \tilde{b}_j$ are partial lifts of the a_j and b_j to positive elements, and $s = \tilde{a}_1 + \tilde{a}_2 + \tilde{b}_1 + \tilde{b}_2$, then down the tower s is eventually invertible since its image downstairs is 1; replace a_j by $s^{-1/2}a_js^{-1/2}$ and b_j by $s^{-1/2}b_js^{-1/2}$ to get lifts exactly satisfying the relation, but a clue to a potential collective problem is that this operation may ruin orthogonality of the a_j or b_j). Note that the relations imply that a_1 and a_2 commute, as do b_1 and b_2. If we set $a = a_1 + a_2$ and $b = b_1 + b_2$, the last relation simply says $b = 1 - a$. As a result, since a commutes with both a_1 and a_2, so does b, and hence b (and therefore also a) commutes with all four generators. Thus a and b are in the center of A.

This C*-algebra can be realized in some different ways. One way of realizing A is that it is the universal C*-algebra generated by the image of a rank one ucp map from $\mathbb{C}^4$ which is the sum of two rank zero maps from $\mathbb{C}^2$. A C*-algebra has nuclear dimension ≤ 1 (one characterization of classifiable simple C*-algebras) if and only if every finite set of elements can be approximated by elements in the image of a rank one map from a finite-dimensional C*-algebra, and this is the simplest nontrivial case of such a map.

To understand the structure of A, consider its irreducible representations. If ρ is an irreducible representation of A on a Hilbert space $\mathcal{H}$, then $\rho(a)$ must be a scalar multiple of 1 since a is in the center of A. If $\rho(a) = t1$ for $0 \leq t \leq 1$, then since $a = a_1 + a_2$ and $a_1 \perp a_2$, it follows that $\rho(a_1) = tp$ and $\rho(a_2) = t(1-p)$ for some projection p. Similarly, $\rho(b) = (1-t)1$ and $\rho(b_1) = (1-t)q$, $\rho(b_2) = (1-t)(1-q)$ for some projection q. We have that $\rho(A)$ is generated by p, q, and 1; since ρ is irreducible, it follows that $\mathcal{H}$ has dimension ≤ 2 (), i.e. A is 2-subhomogeneous.

If $t = 0$, then $\rho(A)$ is generated by q and 1 and is thus commutative, so $q = 0$ or $q = 1$ and we get a one-dimensional representation. There are two of these, ρ_{b1} and ρ_{b2}:

$$\rho_{b1}(a_1) = \rho_{b1}(a_2) = \rho_{b1}(b_2) = 0 \ , \quad \rho_{b1}(b_1) = 1$$

$$\rho_{b2}(a_1) = \rho_{b2}(a_2) = \rho_{b2}(b_1) = 0 \ , \quad \rho_{b2}(b_2) = 1$$

Similarly, if $t = 1$, we get two one-dimensional representations ρ_{a1} and ρ_{a2}:

$$\rho_{a1}(a_2) = \rho_{a1}(b_1) = \rho_{a1}(b_2) = 0 \ , \quad \rho_{a1}(a_1) = 1$$

$$\rho_{a2}(a_1) = \rho_{a2}(b_1) = \rho_{a2}(b_2) = 0 \ , \quad \rho_{a2}(a_2) = 1$$

For $0 < t < 1$, we get four one-dimensional irreducible representations ρ_{11t}, ρ_{12t}, ρ_{21t}, ρ_{22t} corresponding to the cases $p = 0$ or 1 and therefore $q = 0$ or 1:

$$\rho_{11t}(a_1) = t \ , \quad \rho_{11t}(b_1) = 1 - t \ , \quad \rho_{11t}(a_2) = \rho_{11t}(b_2) = 0$$

$$\rho_{12t}(a_1) = t \ , \quad \rho_{12t}(b_2) = 1 - t \ , \quad \rho_{12t}(a_2) = \rho_{12t}(b_1) = 0$$

$$\rho_{21t}(a_2) = t \ , \quad \rho_{21t}(b_1) = 1 - t \ , \quad \rho_{21t}(a_1) = \rho_{21t}(b_2) = 0$$

$$\rho_{22t}(a_2) = t \ , \quad \rho_{22t}(b_2) = 1 - t \ , \quad \rho_{22t}(a_1) = \rho_{22t}(b_1) = 0$$

As $t \to 0$, ρ_{11t} and ρ_{21t} approach ρ_{b1} and ρ_{12t} and ρ_{22t} approach ρ_{b2}; as $t \to 1$ ρ_{11t} and ρ_{12t} approach ρ_{a1} and ρ_{21t} and ρ_{22t} approach ρ_{a2}. Thus the one-dimensional representations form a circle or square in $\text{Prim}(A)$ with four distinguished corner points.

There is also a two-parameter family of two-dimensional irreducible representations of A, corresponding to the cases $0 < t < 1$ and p and q both nontrivial. Up to unitary equivalence we may assume $p = diag(1, 0)$ and

$$q = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

for $0 < \theta < \frac{\pi}{2}$. We thus get a two-dimensional irreducible representation $\rho_{t\theta}$ for each (t, θ), $0 < t < 1$, $0 < \theta < \frac{\pi}{2}$:

$$\rho_{t\theta}(a_1) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} \ , \ \ \rho_{t\theta}(a_2) = \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix}$$

$$\rho_{t\theta}(b_1) = \begin{bmatrix} (1-t)\cos^2\theta & (1-t)\cos\theta\sin\theta \\ (1-t)\cos\theta\sin\theta & (1-t)\sin^2\theta \end{bmatrix} \ , \ \ \rho_{t\theta}(b_2) = \begin{bmatrix} (1-t)\sin^2\theta & -(1-t)\cos\theta\sin\theta \\ -(1-t)\cos\theta\sin\theta & (1-t)\cos^2\theta \end{bmatrix}$$

and up to unitary equivalence these are all the two-dimensional irreducible representations. The topology in Prim(A) on these is the usual topology on $(0,1) \times \left(0, \frac{\pi}{2}\right)$. As $t \to t_0 \in (0,1)$, $\theta \to 0$, $\rho_{t\theta}$ approaches both ρ_{11t_0} and ρ_{22t_0}; as $t \to t_0 \in (0,1)$, $\theta \to \frac{\pi}{2}$, $\rho_{t\theta}$ approaches both ρ_{12t_0} and ρ_{21t_0}. As $t \to 0$, $\rho_{t\theta}$ approaches both ρ_{b1} and ρ_{b2} regardless of θ, and as $t \to 1$ $\rho_{t\theta}$ approaches both ρ_{a1} and ρ_{a2}. Thus Prim(A) is T_1 but not Hausdorff: it looks like the open unit disk $\mathbb{D}$ with a circle wrapped twice-around on the boundary. It can be shown that A is isomorphic to a C*-subalgebra of $C(\bar{\mathbb{D}}, \mathbb{M}_2)$ where values on the boundary are "diagonal" with respect to an orthonormal basis which rotates around the circle (details are left to the interested reader).

Because Prim(A) is two-dimensional, A is *not* semiprojective ().

VI.2.5.7. There are more bad apples which are not commutation relations, at least not in any obvious way. For example, there is an action α of $\mathbb{Z}_2$ on the Cuntz algebra O_2 such that $A = O_2 \rtimes_\alpha \mathbb{Z}_2$ is a UCT Kirchberg algebra which does not have finitely generated K-theory, hence is not semiprojective. Note that $A \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \cong M_2(O_2) \cong O_2$ is semiprojective. See [**?**] for details of this example.

Although there is no obvious presentation for the crossed product (there are three obvious generators, or five, with some obvious relations but not enough; using the classification theorem there is an explicit non-finite presentation), one might expect that a crossed product of a semiprojective C*-algebra by a finite cyclic group should be semiprojective, since it is "half" the process of forming a finite matrix algebra, which preserves semiprojectivity (although such crossed products can fail to preserve other natural things such as real rank zero (), and possibly do not even necessarily stay in the UCT class ()).

Inductive Limits

VI.2.5.8. It seems unlikely (and in fact is usually false, as easy examples like $\mathbb{K}$ show) that an inductive limit of semiprojective C*-algebras is semiprojective. For one thing, if $A = \lim_{\to} A_k$, (B, J_n, J) is a tower, and $\phi : A \to B/J$ a homomorphism, the restriction of ϕ to A_k can be partially lifted, but the n might have to be increased as k increases so there might not be a single n that works for all k. But the situation is worse than that. If this were the only problem, one would expect that an inductive limit of projective C*-algebras should be projective. But there is an additional problem: to illustrate, suppose $A_1 \subseteq A_2$ and both are projective, ϕ_2 is a homomorphism from A_2 to B/J for some B and J, and $\phi_1 = \phi_2|_{A_1}$. Then ϕ_1 lifts to a homomorphism ψ_1 from A_1 to B, but such a lift may not extend to a lift of ϕ_2 from A_2 to B. Of course, in this case there *exists* a lift ψ_1 of ϕ_1 which extends to A_2, obtained by first lifting ϕ_2 and restricting to A_1, but this procedure does not work in an inductive limit since there is no place to start. In fact, an inductive limit of projective C*-algebras is not necessarily even semiprojective: for example, $C\mathbb{M}_k$ is projective for each k (), and $C\mathbb{K}$ is an inductive limit of these in an obvious way, but $C\mathbb{K}$ is not semiprojective by VI.2.4.19. since it has no nonzero finite-dimensional quotients. There are lots of commutative examples: any (compact metrizable) space is an inverse limit of ANRs (and ARs if the space is homotopically trivial). See () for a discussion of which simple C*-algebras are inductive limits of semiprojectives.

VI.2.6. Tastier High-Hanging Fruit

Fruit from higher in the tree is often tastier, but requires much more effort to get it. We now discuss some additional results about semiprojective C*-algebras which are harder to prove.

VI.2.7. Amalgamated Free Products

as an amalgamated free product

$$[C_0((0,1]) \oplus C_0((0,1])] \star_{C_0((0,1))} [C_0([0,1)) \oplus C_0([0,1))]$$

where $C_0((0,1]) = \{f \in C([0,1]) : f(0) = 0\}$, $C_0([0,1)) = \{f \in C([0,1]) : f(1) = 0\}$, and

$$C_0((0,1)) = \{f \in C([0,1]) : f(0) = f(1) = 0\}$$

is embedded in both direct sums as $\{(f,f) : f \in C_0((0,1))\}$. The generators a_1 and a_2 correspond to the elements $(t,0)$ and $(0,t)$ of $C_0((0,1]) \oplus C_0((0,1])$, where t denotes the function $f(t) = t$, and b_1 and b_2 to the elements $(1-t,0)$ and $(0,1-t)$ of $C_0([0,1)) \oplus C_0([0,1))$ where $1-t$ denotes the function $g(t) = 1-t$.

Chapter VII

Regularity

One of the main themes behind the classification program as it has developed is that there is a strict dichotomy or "0-1 law": there is a class of separable simple nuclear C*-algebras which are very well behaved and which can be classified by a relatively simple invariant, and a menagerie of other separable simple nuclear C*-algebras which are very badly behaved and defy reasonable description or classification (not to mention a huge unwashed horde of C*-algebras which fail to be separable, simple, and/or nuclear). To paraphrase the nursery rhyme, when a (separable nuclear) simple C*-algebra is good (regular), it is very, very good, and when it is bad (irregular) it is horrid.

Regularity comes in many forms:

Finite nuclear dimension

Nuclear dimension ≤ 1

$\mathcal{Z}$-stability

Strict comparability

Well-behaved Cuntz semigroup

all of which are equivalent (there are other actual or potential forms too). Significantly, and fortunately, regularity is automatic in many standard constructions and examples, so the regular simple C*-algebras form a vast subclass including at least a large fraction of the simple C*-algebras which arise "naturally" in the theory or in applications; in fact, one has to work pretty hard to construct nonregular ones.

In this chapter we will examine the various aspects of regularity and prove the equivalences. This will provide the natural class to be classified.

Chapter VIII

Stability, Finite-Dimensionality, and Classification

VIII.1. $\mathcal{Z}$-Stability

VIII.1.1. Properties of $\mathcal{Z}$

VIII.1.2. $\mathcal{Z}$-Stability, Rational Stability, and Slow Dimension Growth

VIII.2. Comparability

VIII.2.1. Comparability of Projections

VIII.2.2. Comparability of Positive Elements

VIII.3. Finite Decomposition Rank and Finite Nuclear Dimension

VIII.3.1. Decomposition Rank vs. Nuclear Dimension

VIII.4. The Role of Quasidiagonality

VIII.4.1. Quasidiagonal Traces

VIII.5. Semiprojectivity

This section is very speculative at present.

VIII.5.1. Semiprojective C*-Algebras

VIII.5.2. Approximate Semiprojectivity

VIII.5.2.1. DEFINITION. A (separable) nuclear C*-algebra A is *approximately semiprojective* if, for any $x_1, \dots, x_n \in A$ and $\epsilon > 0$, there is a nuclear semiprojective C*-algebra D and a homomorphism $\phi : D \to \mathcal{A}$ (not necessarily injective), and elements $z_1, \dots, z_n \in D$, with

$$\|x_k - \phi(z_k)\| < \epsilon$$

for $1 \leq k \leq n$.

This might not be the best terminology, since the definition might be better with the "nuclear" removed; but then there would be an ambiguity if A is nuclear whether the D should be required to be nuclear. Maybe the definition as written should be something like *nuclearly approximately semiprojective*, but I haven't found a term I am happy with. I think this is the right definition, but I don't know that it is phrased in optimal form; there may be an equivalent phrasing which is better.

VIII.5.2.2. The class $\mathcal{A}$ of approximately semiprojective C*-algebras is more flexible than the class of (nuclear) semiprojective C*-algebras:

$\mathcal{A}$ contains all nuclear semiprojective C*-algebras.

$\mathcal{A}$ is closed under quotients.

$\mathcal{A}$ is closed under inductive limits.

$\mathcal{A}$ is closed under stable isomorphism (Morita equivalence).

It is possible that $\mathcal{A}$ is also closed under extensions, but this is unclear. It is certainly *not* closed under tensor products. It is also unclear whether $\mathcal{A}$ is the smallest class containing the semiprojectives which is closed under these operations.

VIII.5.2.3. **Conjecture:** Let A be a separable nuclear C*-algebra. Then A is approximately semiprojective if and only if it has nuclear dimension ≤ 1.

Note that to prove one direction, it suffices to show that every nuclear semiprojective C*-algebra has nuclear dimension ≤ 1.

VIII.5.2.4. There is rather scanty evidence for this conjecture (either direction!), but there is some:

(i) If X is a compact metrizable space, then $C(X)$ has nuclear dimension ≤ 1 if and only if $dim(X) \leq 1$ (). $C(X)$ is semiprojective if and only if X is an ANR of dimension ≤ 1 (). If $dim(X) \leq 1$, then X can be written as an inverse limit of polyhedra of dimension ≤ 1, and conversely (), thus $C(X)$ is an inductive limit of semiprojectives if and only if $dim(X) \leq 1$. So the conjecture is true for unital commutative C*-algebras. This can be extended to the nonunital case, and more generally to subhomogeneous C*-algebras ().

(ii) At least almost all classifiable simple C*-algebras are approximately semiprojective: all purely infinite ones are (), and all stably finite ones with torsion-free K_0 are too (). In fact, all these are inductive limits of a sequence of semiprojective C*-algebras with injective connecting maps. The suspect ones are the classifiable stably finite algebras with torsion in K_0: two-dimensional building blocks must be used to construct them as ASH algebras, which very likely means they cannot be written as inductive limits of semiprojectives with injective connecting maps. They could very well nonetheless be approximately semiprojective. Perhaps they are at least *tracially* approximately semiprojective, a possible variation of the conjecture.

At least there is no separable nuclear C*-algebra for which the conjecture is known to fail. We do not yet know enough about nuclear semiprojective C*-algebras to confidently evaluate whether the conjecture is reasonable.

VIII.5.2.5. If the conjecture is true, then the classifiable C*-algebras are exactly the separable simple unital approximately semiprojective C*-algebras.

VIII.5.2.6. Approximately semiprojective C*-algebras have a slightly relaxed version of the rigidity which semiprojective C*-algebras have (), and thus if the conjecture holds (even just in the simple unital case), there is a possibility of a simplified proof of the classification theorem.

VIII.6. Putting the Pieces Together

VIII.6.1. Equivalence of the Characterizations

VIII.6.1.1. $\mathcal{Z}$-Stable C*-algebras are the only ones for which there is hope of a classification, since if A is a C*-algebra, then A and $A \otimes \mathcal{Z}$ have the same Elliott invariant (up to perforation), and $A \otimes \mathcal{Z}$ is $\mathcal{Z}$-stable. Elliott has described $\mathcal{Z}$-stable C*-algebras as being "tenderized" (this was apparently originally an inadvertent misprint of "tensorized," i.e. "tensored with $\mathcal{Z}$," but is an appropriately descriptive term). Thus the classification is for tenderized separable simple UCT C*-algebras.

VIII.6.2. The Classification Theorem

Here is the full statement of the Classification Theorem in the unital case, including the characterization of the invariant. This is of course a special case of the general Classification Theorem (), but the statement can be made somewhat more cleanly in the unital case.

VIII.6.2.1. Theorem. [Classification Theorem, Unital Case] Let $\mathcal{C}_1$ be the class of all regular infinite-dimensional, separable, simple, unital, UCT C*-algebras. If $A \in \mathcal{C}_1$, set

$$Ell(A) = (K_0(A), K_0^+(A), [1_A], \mathcal{T}(A), \rho_A, K_1(A))$$

where $K_*(A) = (K_0(A), K_1(A))$ is the K-theory of A, $K_0^+(A)$ is the image of $V(A)$ in $K_0(A)$, $[1_A]$ is the class of the unit of A in $K_0^+(A)$, $\mathcal{T}(A)$ is the set of (normalized) traces on A, and $\rho_A : K_0(A) \to \mathrm{Aff}(\mathcal{T}(A))$ is the pairing homomorphism (where $\mathrm{Aff}(\mathcal{T}(A))$ is taken to be $\{0\}$ if $\mathcal{T}(A) = \emptyset$). The class $\mathcal{C}_1$ is called the *(unital) classifiable class* and $Ell(A)$ is called the *Elliott invariant* of A. Then:

(i) (*Properties of the Invariant*) If $A \in \mathcal{C}_1$, then $Ell(A)$ has the following properties:

(a) $K_0(A)$ and $K_1(A)$ are countable abelian groups.

(b) $\mathcal{T}(A)$ is a metrizable Choquet simplex (possibly empty).

(c) $K_0^+(A) = \{0\} \cup \{x \in K_0(A) : \rho_A(x) \text{ is strictly positive}\}$ if $\mathcal{T}(A) \neq \emptyset$, and $K_0^+(A) = K_0(A)$ if $\mathcal{T}(A) = \emptyset$.

(d) $\rho_A([1_A])$ is the constant function 1 if $\mathcal{T}(A) \neq \emptyset$.

(ii) (*Characterization of the Invariant*) Let $(G_0, G_0^+, u, \Delta, \rho, G_1)$ be a tuple such that

(a) G_0 and G_1 are countable abelian groups, and $u \in G_0$.

(b) Δ is a metrizable Choquet simplex (possibly empty).

(c) $\rho : G_0 \to \mathrm{Aff}(\Delta)$ is a homomorphism, and $G_0^+ = \{0\} \cup \{x \in G_0 : \rho(x) \text{ is strictly positive}\}$ if $\Delta \neq \emptyset$, and $G_0^+ = G_0$ if $\Delta = \emptyset$.

(d) $\rho(u)$ is the constant function 1 if $\Delta \neq \emptyset$.

Then there is an $A \in \mathcal{C}_1$ with $Ell(A) \cong (G_0, G_0^+, u, \Delta, \rho, G_1)$.

(iii) (*Completeness of the Invariant*) If $A, B \in \mathcal{C}_1$ and $\phi : Ell(A) \to Ell(B)$ is an isomorphism, then there is an isomorphism from A to B inducing ϕ. In particular, if $Ell(A) \cong Ell(B)$, then $A \cong B$. Thus the C*-algebra in (ii) is unique up to isomorphism.

(iv) (*Characterization of the Class* $\mathcal{C}_1$) Let A be a separable simple unital C*-algebra satisfying the UCT. The following are equivalent:

(a) $A \in \mathcal{C}_1$, i.e. A is regular.

(b) A has finite nuclear dimension.

(c) A has nuclear dimension ≤ 1.

(d) A is $\mathcal{Z}$-stable.

(e) A has strict comparability of positive elements.

(v) (*Structure of Classifiable C*-Algebras*)

(a) If $A \in \mathcal{C}_1$, then A is either stably finite or purely infinite.

(b) If A is a C*-algebra, the following are equivalent:

(1) $A \in \mathcal{C}_1$ and A is stably finite.

(2) $A \in \mathcal{C}_1$ and $\mathcal{T}(A) \neq \emptyset$.

(3) $A \in \mathcal{C}_1$ and $sr(A) = 1$.

(4) A is infinite-dimensional, separable, simple, unital, UCT, and has finite decomposition rank.

(5) A is a simple unital ASH algebra with slow dimension growth.

(6) A is a simple unital ASH algebra with base spaces of dimension ≤ 2.

(c) If A is a C*-algebra, the following are equivalent:

(1′) $A \in \mathcal{C}_1$ and A is purely infinite.

(2′) $A \in \mathcal{C}_1$ and $\mathcal{T}(A) = \emptyset$.

(3′) $A \in \mathcal{C}_1$ and $sr(A) > 1$.

(4′) $A \in \mathcal{C}_1$ and $sr(A) = \infty$.

(5′) A is a UCT Kirchberg algebra (separable, UCT, and purely infinite).

(d) If $A \in \mathcal{C}_1$, the following are equivalent:

(7) A has real rank zero.

(8) A has (SP) and (PT).

(9) The range of $\rho_A : K_0(A) \to \text{Aff}(\mathcal{T}(A))$ is uniformly dense.

(10) A is either purely infinite or an AH algebra with base spaces of dimension ≤ 3 and real rank zero.

(e) If A is an infinite-dimensional separable simple unital C*-algebra, the following are equivalent:

(11) A is an AF algebra.

(12) A has nuclear dimension zero.

(13) A has decomposition rank zero.

(14) $A \in \mathcal{C}_1$, A is stably finite, $rr(A) = 0$, $K_0(A)$ is torsion-free, and $K_1(A) = 0$.

(11)–(13) are equivalent for any separable C*-algebra A; the other hypotheses are only needed for equivalence with (14).

VIII.6.2.2. By an isomorphism of tuples $(G_0, G_0^+, u, \Delta, \rho, G_1)$ and $(G_0', {G_0'}^+, u', \Delta', \rho', G_1')$, we mean group isomorphisms $\phi_i : G_i \to G_i'$ $(i = 0, 1)$ with $\phi_0(u) = u'$ and an affine homeomorphism $\psi : \Delta \to \Delta'$ inducing $\hat{\psi} : \text{Aff}(\Delta') \to \text{Aff}(\Delta)$ with $\rho = \hat{\psi} \circ \rho' \circ \phi_0$ (so automatically $\phi_0(G_0^+) = {G_0'}^+$). An isomorphism from A to B induces an isomorphism from $Ell(A)$ to $Ell(B)$ in the obvious way.

VIII.6.2.3. Of course, (iii) is the heart of the Classification Theorem. But the other parts (some of which themselves are quite nontrivial) are more than just ancillary details and are worth including in the statement to get the full classification picture.

VIII.6.2.4. This theorem is sharp. The results of (i) are true in greater generality (much greater for all but (c)), but the overall theorem cannot, with one possible exception, be extended or generalized without substantial modification of the statement. The only exception is that it may eventually be shown that the UCT hypothesis is automatic and can thus be eliminated from the statement.

The statement can be modified by replacing the finite nuclear dimension hypothesis with an equivalent one such as $\mathcal{Z}$-stability, although a nuclearity hypothesis will then have to be added if the UCT hypothesis is eventually eliminated (note that nuclearity is entailed by finite nuclear dimension but not $\mathcal{Z}$-stability).

VIII.6.2.5. The $K_0^+(A)$ (or G_0^+ in the abstract case) can actually be eliminated from the invariant, since it is completely determined from the rest of the invariant by (c). But it is customary to include it in the Elliott invariant with its definition as the image of $V(A)$, with the nontrivial result that it is equal to the set characterized in (ic). In the Elliott invariant, the ρ can also be replaced by the map θ as in III.4.1.2.; it is a matter of taste how the invariant is written (the overall statement seems cleaner with ρ).

VIII.6.3. Classification of Homomorphisms

The Classification Theorem says that the Elliott invariant functor from $\mathcal{C}$ to $\mathcal{E}$ is a bijection on isomorphism classes of objects. But this does not quite mean it is a category equivalence; there must be a characterization of when morphisms in $\mathcal{C}$ induce the same morphism in $\mathcal{E}$. It turns out that a more delicate invariant is needed.

VIII.6.4. The Toms-Winter Conjecture

VIII.6.4.1. Theorem.(?) Let A be a separable simple unital nuclear C*-algebra (UCT?) TFAE:

(i) A has finite nuclear dimension.

(ii) A has nuclear dimension ≤ 1.

(iii) A is $\mathcal{Z}$-stable.

(iv) A has comparability of positive elements.

(v) A is purely infinite, or ASH with base spaces of dimension ≤ 2.

(vi) A is approximately semiprojective.

If A has (SP), these are also equivalent to:

(vii) A is approximately divisible.

VIII.6.5. The Real Rank Zero Case

VIII.6.5.1. Theorem. Let A be a separable simple unital UCT C*-algebra of real rank zero. If A is purely infinite or of stable rank 1, then A satisfies the conditions of () and is thus in the classifiable class.

VIII.6.5.2. Is every separable simple unital UCT C*-algebra of real rank zero in the classifiable class (i.e. either purely infinite or stable rank 1)?

VIII.7. Classification of Nonunital Simple C*-Algebras

Classification of nonunital C*-algebras satisfying the hypotheses of the Classification Theorem is a somewhat subtle and thorny matter, but can be done with reasonable extension of the previous methods.

VIII.7.1. The Non-Projectionless Case

Some of the classification in the nonunital case can be quickly reduced to the unital classification, using the observation that if A is a separable simple C*-algebra containing a nonzero projection p, then A is stably isomorphic to pAp.

The Infinite Case

The "infinite" nonunital case can be polished off by the following extension of this observation from [BC82] (rephrased in an appropriate form for present purposes):

VIII.7.1.1. Theorem. Let A be a separable simple C*-algebra. If A has no nonzero (bounded or unbounded) dimension function, or equivalently no nonzero (bounded or unbounded) quasitrace, then A is stably isomorphic to a properly infinite (separable simple) unital C*-algebra.

VIII.7.1.2. Suppose A satisfies the hypotheses, and is in the UCT class and is $\mathcal{Z}$-stable. Then A is stably isomorphic to a properly infinite separable simple unital $\mathcal{Z}$-stable UCT C*-algebra, i.e. a UCT Kirchberg algebra. We then use the following result, which is a simple consequence of the fact that a purely infinite (simple unital) C*-algebra has real rank zero ():

VIII.7.1.3. Proposition. Let A be a separable C*-algebra which is stably isomorphic to a purely infinite (simple unital) C*-algebra. Then A is either unital or stable.

Combining with the previous paragraph, we obtain:

VIII.7.1.4. Theorem. Let A be a separable simple $\mathcal{Z}$-stable UCT C*-algebra with no nonzero (bounded or unbounded) dimension function, or equivalently no nonzero (bounded or unbounded) trace. Then A is either a UCT Kirchberg algebra (if unital) or a stabilized UCT Kirchberg algebra (if nonunital).

We then get a deceptively simple classification theorem:

VIII.7.1.5. Corollary. Let A and B be nonunital separable simple $\mathcal{Z}$-stable UCT C*-algebras with no nonzero (bounded or unbounded) dimension function, or equivalently no nonzero (bounded or unbounded) trace. Then $A \cong B$ if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$, i.e. the pair $(K_0(A), K_1(A))$ is a complete isomorphism invariant for A among such C*-algebras. $K_0(A)$ and $K_1(A)$ can be any countable abelian groups, i.e. if G_0 and G_1 are any countable abelian groups, there is such an A, unique up to isomorphism, with $K_0(A) \cong G_0$ and $K_1(A) \cong G_1$.

Note that in the UCT class, isomorphism of K-theory is the same as KK-equivalence, so if A and B satisfy the hypotheses, then $A \cong B$ if and only if A and B are KK-equivalent.

The Stably Finite Case

Classification of nonunital separable simple $\mathcal{Z}$-stable C*-algebras with a nonzero trace and which are stably isomorphic to a unital C*-algebra (e.g. contain a nonzero projection) is more complicated, but can also be reduced to the unital case; it amounts to giving an explicit description, for any classifiable unital A, of all C*-algebras stably isomorphic to A, or equivalently of all hereditary C*-subalgebras of $A \otimes \mathbb{K}$.

VIII.7.1.6. If A has real rank zero, this can be rather efficiently done by just considering upward directed hereditary generating subsets of the positive cone of K_0, which amounts to simply considering strictly increasing sequences in the positive cone. The main subtlety comes in identifying when two such subsets are conjugate under an order-automorphism of $K_0(A)$. The AF examples from () illustrate the process.

VIII.7.2. The Stably Projectionless Case

The principal difficulty in the nonunital classification problem (beyond the unital classification theorem) is classifying (separable) stably projectionless simple C*-algebras in a suitable class (e.g. UCT, $\mathcal{Z}$-stable). Such algebras necessarily have a nonzero trace, either bounded or unbounded (VIII.7.1.1.). They can nonetheless have nontrivial K_0 (as well as nontrivial K_1).

VIII.7.3. Classification Via the Cuntz Semigroup

VIII.7.3.1. There is a very clean way of looking at the Classification Theorem using the Cuntz semigroup. In fact, the classifiable algebras are exactly the separable simple UCT C*-algebras with "tractable" Cuntz semigroup, and the Cuntz semigroup nicely encodes almost all of the Elliott invariant.

VIII.7.3.2. Definition. Let A be a C*-algebra. The *Cuntz invariant* of A is the triple

$$Cuntz(A) = (\mathrm{Cu}(A), K_1(A), \Sigma_c)$$

where $\mathrm{Cu}(A)$ is the Cuntz semigroup () of $A \otimes \mathbb{K}$, $K_1(A)$ is the usual K_1-group of A, and Σ_c is the *scale* of $\mathrm{Cu}(A)$, the set of elements of $\mathrm{Cu}(A)$ coming from (positive) elements of A.

The Elliott invariant of A can be recovered from the Cuntz invariant of A in some cases (perhaps all cases if the Cuntz invariant is modified a little, cf. VIII.7.3.5.):

Classifiable simple unital C*-algebras are precisely ones with nicely behaved Cuntz invariant:

VIII.7.3.3. The Cuntz invariant, unlike the Elliott invariant, even distinguishes $\mathbb{C}$ from the Jiang-Su algebra $\mathcal{Z}$, so the "infinite-dimensional" restriction can be removed from the Classification Theorem.

The Classification Theorem (unital stably finite version) can be thus rephrased:

VIII.7.3.4. Theorem. The Cuntz invariant is a complete isomorphism invariant for separable simple stably finite UCT C*-algebras of finite nuclear dimension which are unital, or more generally contain a nonzero projection: if A and B are two such algebras and $Cuntz(A) \cong Cuntz(B)$, then $A \cong B$.

VIII.7.3.5. In the purely infinite case, the Cuntz invariant should be modified slightly. If A is purely infinite, then $\mathrm{Cu}(A)$ consists precisely of three elements: $[0]$, $[1]$, and $[a]$, where $[0]$ is the class of 0, $[1]$ is the class consisting of all nonzero projections (including 1_A) and, more generally, of all nonzero well-supported elements of A, and $[a]$ is the class of all elements which are not well supported. These are ordered $[0] < [1] < [a]$, and the addition is $x + y = \max(x, y)$. $V_f(A)$ () collapses to $[1]$ in $\mathrm{Cu}(A)$, and thus $\mathrm{Cu}(A)$ does not encode $K_0(A)$. So $[1]$ should be replaced by $V_f(A)$ in $\mathrm{Cu}(A)$, with the obvious modification of the addition. Also, Σ_c is all of $\mathrm{Cu}(A)$ in this case, so is not helpful in distinguishing stably isomorphic algebras; it should instead be $[1_A]$ in the modified $\mathrm{Cu}(A)$ if A is unital, and $[a]$ if A is nonunital (stable). (We could in general take Σ_c to be the class of a strictly positive element of A if A is separable or, more generally, σ-unital; this is well defined since all strictly positive elements of A are Cuntz equivalent.) With these modifications, the classification of VIII.7.3.4. applies also in the purely infinite case, i.e. "stably finite" can be removed from the statement.

If A is neither stably finite nor purely infinite, e.g. in Rørdam's example (), $\mathrm{Cu}(A)$ should be modified similarly, but it will be more complicated than in the purely infinite case, and will distinguish A from the UCT Kirchberg algebra with the same Elliott invariant.

VIII.7.3.6. The Cuntz invariant is also inadequate as it stands in the stably projectionless case, since it does not encode K_0 or the pairing. Thus it will have to be augmented by

We may make a bold conjecture (for which there is scant evidence but no counterexample at present):

VIII.7.3.7. Conjecture. The Cuntz invariant (suitably modified) is a complete isomorphism invariant for separable simple nuclear (or at least UCT) C*-algebras.

The conjecture could even more boldly remove the "simple," since the Cuntz invariant of A also encodes the ideal structure of A.

VIII.7.3.8. This conjecture, if it becomes a Theorem, would be a spectacular generalization of the current Classification Theorem, at least in theory. However, it might not be much of an advance in a practical sense, since the Cuntz semigroup is essentially impossible to compute for C*-algebras much beyond the currently classified ones. One great virtue of the current Classification Theorem is that it avoids explicit use of the Cuntz semigroup, and is expressed in terms of an invariant which is actually computable in many cases.

VIII.7.3.9. The Cuntz invariant of A could be refined by considering the Cuntz semigroup of $A \otimes B$ for some B which would encode $K_1(A)$. This has been considered for K-theory by taking $B = C(\mathbb{T})$, since $K_0(A \otimes C(\mathbb{T})) \cong K_0(A) \oplus K_1(A)$, and the ordering on $K_0(A \otimes C(\mathbb{T}))$ is more delicate than just the strict ordering in general. However, the Cuntz semigroup of $A \otimes C(\mathbb{T})$ is likely pathological even if A is classifiable. Perhaps replacing $C(\mathbb{T})$ with a stably finite classifiable unital C*-algebra B with $K_0(B) = K_1(B) = \mathbb{Z}$ and unique trace would work better.

VIII.7.3.10. The class of nuclear C*-algebras is the largest class of C*-algebras for which there could be any reasonable hope of such a classification. For example, all Type II_1 factors have the same Cuntz invariant, as do all (countably decomposable) Type III factors (these are admittedly not separable). Also, the algebra $C^*_r(G)$ of I.6.5.3., or at least $C^*_r(G) \otimes \mathcal{Z}$, likely has the same Cuntz invariant as the Jiang-Su algebra $\mathcal{Z}$, and $A \otimes C^*_r(G)$ probably has the same Cuntz invariant as A for any classifiable algebra A.

VIII.8. What Next?

Nonsimple case, real rank zero, homotopy and shape equivalence, crossed products, exact, ...

Existence of Pathological Examples

VIII.8.0.11. There remain several potential types of (separable) simple unital UCT C*-algebras not known to exist, but not ruled out (as of this writing), summarized here: Does there exist such an A such that

(i) A is stably finite, has real rank zero, but not stable rank 1? (Such an example could conceivably exist even in the AH case without slow dimension growth.) An example would have to have perforated K_0, or else fail to have cancellation, i.e. the pathology would be detectible in the semigroup $V(A)$.

(ii) A has real rank zero, and is finite but not stably finite?

(iii) Every nonzero projection in A is infinite, but A is not purely infinite (i.e. A contains a nonzero x such that the hereditary C*-subalgebra A_x is projectionless, or equivalently A does not have real rank zero)?

(iv) A is stably finite but not ASH?

(v) A is stably finite but not NF (i.e. not quasidiagonal)?

(vi) A is not isomorphic to its opposite algebra A^{op}? (This is not possible in the purely infinite case by the classification, but potentially possible in the stably finite case, even the AH or ASH case without slow dimension growth.)

(vii) A has stable rank 1 and $\mathcal{T}(A)$ is a Bauer simplex, but A is not classifiable (i.e. not $\mathcal{Z}$-stable)? See [**?**].

We of course also wonder whether there is a separable simple nuclear C*-algebra not satisfying the UCT. If there is any separable nuclear C*-algebra not satisfying the UCT, there will be a purely infinite (simple) unital one with trivial K-theory (a version of O_2 which is not KK-contractible). Is there one with finite nuclear dimension?

Automorphisms and Group Actions on Classifiable C*-Algebras

Subalgebras of Finite Index

One could ask, in analogy with the problem of classifying subfactors of finite index in the hyperfinite II_1 factor (or other AFD factors), to classify simple C*-subalgebras of finite index in classifiable simple C*-algebras. In addition to technical problems in even defining an index for a C*-subalgebra, the difficulties are much greater than in the factor case (which is already hard enough!). For one thing, one cannot hope to get very far until the UCT problem is solved: if the UCT fails, there is a simple unital C*-subalgebra of O_2 of index 2 which does not satisfy the UCT (index should at least be defined for crossed products by finite groups). And even in the simplest cases, the variety is greater: there are simple index 2 subalgebras of the CAR algebra which are not AF algebras and do not even have real rank zero ([Bla90b], [**?**]). At least a C*-subalgebra of finite index in a classifiable simple C*-algebra should have finite nuclear dimension (?), hence should be classifiable if it satisfies the UCT.

DRAMATIS PERSONÆ

Act I (1928 – ca. 1978)

Huzihiro Araki
Ola Bratteli
Larry Brown
John Bunce
Man-Duen Choi
Alain Connes
Joachim Cuntz
James Deddens
Jacques Dixmier
Edward Effros
George Elliott
Israel Gelfand
James Glimm
Wolfgang Krieger
Dusa McDuff
Frank Murray
Mark Naimark
William Paschke
Robert Powers
Masamichi Takesaki
John von Neumann
James Woods

Act II (ca. 1978 – ca. 1998)

Joel Anderson
Bruce Blackadar
Larry Brown
Man-Duen Choi
Joachim Cuntz
Marius Dadarlat
Edward Effros
Søren Eilers
George Elliott
David Evans
Guihua Gong
Kenneth Goodearl
Uffe Haagerup
David Handelman
Richard Herman
Eberhard Kirchberg
Alexander Kumjian
Liangqing Li
Huaxin Lin
Karen Egede Nielsen
William Paschke
Gert Pedersen
N. Christopher Phillips
Mihai Pimsner
Sorin Popa
Ian Putnam
Marc Rieffel
Mikael Rørdam
Jonathan Rosenberg
Chao-Liang Shen
Christian Skau
Hongbing Su
Klaus Thomsen
Jesper Villadsen
Dan Voiculescu

Act III (ca. 1998 – ca. 2015)

Nathanial Brown
Alin Ciuperca
Marius Dadarlat
Søren Eilers
George Elliott
Guihua Gong
Ilan Hirshberg
Xinhui Jiang
Eberhard Kirchberg
Huaxin Lin
Hiroki Matui
Zhuang Niu
Narutaka Ozawa
Francesc Perera
Shaloub Razak
Leonel Robert
Mikael Rørdam
Yasuhiko Sato
Christopher Schafhauser
Hongbing Su
Aaron Tikuisis
Andrew Toms
Stuart White
Wilhelm Winter
Joachim Zacharias

Act IV (ca. 2012 – ca. 2025)

Joan Bosa
José Carrión
Jorge Castillejos
Marius Dadarlat
George Elliott
Samuel Evington
James Gabe
Eusebio Gardella
Guihua Gong
Bhishan Jacelon
David Kerr
Eberhard Kirchberg
Huaxin Lin
Hiroki Matui
Zhuang Niu
Francesc Perera
Leonel Robert
Mikael Rørdam
Luis Santiago
Yasuhiko Sato
Christopher Schafhauser
Aaron Tikuisis
Stuart White
Wilhelm Winter

Plus numerous cameos by other players. [Some names undoubtedly need to be added, especially in Act IV.]

Dates of the acts are approximate, and correspond to the eras described in (). Assignment of names to acts are also somewhat arbitrary, and a few names appear more than once when they were major players in different acts.

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