Operator Algebras
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Errata and Addenda

p. XII line -5: since

p. 1-2: I blew this simple formula: should be $\alpha = -\langle \xi, \eta \rangle / \langle \eta, \eta \rangle$.

p. 2 I.1.1.4: The Riesz-Fischer Theorem is often stated this way today, but neither Riesz nor Fischer (who worked independently) phrased it in terms of completeness of the orthogonal system $\{e^{int}\}$. If $[a, b]$ is a bounded interval in $\mathbb{R}$, in modern language the original statement of the theorem was that $L^2([a, b])$ is complete and abstractly isomorphic to l^2.

According to [**?**, p. 385], the name "Hilbert space" was first used in 1908 by A. Schönflies, apparently to refer to what we today call l^2. Von Neumann used the same name for Hilbert spaces in the modern sense (complete inner product spaces), which he defined in 1928.

p. 3 line -6: At the end of the line, 2ϵ should be 4ϵ.

p. 3 I.1.2.3: The statement that a dense subspace of a Hilbert space $\mathcal{H}$ contains an orthonormal basis for $\mathcal{H}$ can be false if $\mathcal{H}$ is nonseparable. In fact, I. Farah (private communication) has shown that a Hilbert space of dimension $2^{\aleph_0}$ has a dense subspace which does not contain any uncountable orthonormal set. A similar example was obtained by Dixmier [**?**].

p. 8-9 I.2.4.3(i): Some of the statements on p. 9 can be false if the measure space is not σ-finite.

p. 13: add after I.2.6.16:
I.2.6.17. If X is a compact subset of $\mathbb{C}$ not containing 0, and $k \in \mathbb{N}$, there is in general no bound on the norm of T^{-1} as T ranges over all operators with $\|T\| \leq k$ and $\sigma(T) \subseteq X$. For example, let $S_n \in \mathcal{L}(l^2)$ be the truncated shift:

$$S_n(\alpha_1, \alpha_2, \dots) = (0, \alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots)$$

and let $T_n = I - S_n$. $\|S_n\| = 1$, so $\|T_n\| \leq 2$ for all n. Since S_n is nilpotent, $\sigma(S_n) = \{0\}$, so $\sigma(T_n) = \{1\}$ for all n. T_n is invertible, with $T_n^{-1} = I + S_n + S_n^2 + \cdots + S_n^n$, and $\|T_n^{-1}\xi_1\| = \sqrt{n+1}$, so $\|T_n^{-1}\| \geq \sqrt{n+1}$.

If T is restricted to normal operators, then there is a bound: if T is an invertible normal operator, then $\|T^{-1}\| = d^{-1}$, where d is the distance from 0 to $\sigma(T)$. This follows immediately from II.1.6.3 and the elementary fact that $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$ (cf. II.1.5.2(i)).

p. 51: Another general operator algebra reference is [**?**]. Another good reference for real C*-algebras is [**?**].

p. 52 lines 17-21: "... the C*-axiom, along with submultiplicativity of the norm, implies that ... It turns out that this weaker axiom, along with the Banach algebra axioms, also implies ..."

p. 58 II.1.6.6: Reference should be II.2.2.10 (see addendum to p. 61).

p. 58: Add at end of paragraph following II.1.6.6:

The adjoint operation on a C*-algebra is also completely determined by the algebra structure and norm (see the addendum to p. 137).

p. 59 end of II.1: Somewhere in the book the (fairly obvious) fact should be explicitly recorded that a C*-algebra is separable if and only if it is countably generated as a C*-algebra. Here is as good a place as any to insert it as II.1.6.9; another possibility is as II.1.1.5 on p. 53.

p. 61: Add after II.2.2.9:

II.2.2.10. Corollary. If A is a C*-algebra with respect to a norm $\|\cdot\|$, then there is no other C*-norm $\|\cdot\|'$ on A (with the same algebraic structure and involution), even incomplete.

Proof: Let B be the completion of A with respect to $\|\cdot\|'$. Then B is a C*-algebra, and the natural injective *-homomorphism from $(A, \|\cdot\|)$ to B is isometric.

p. 61 II.2.3.1: This proof is incomplete. Here is a better argument, also based on II.2.2.4. Let x be a normal element in a C*-algebra A, which we may assume to be unital. Then $C^*(x,1) \cong C(Y)$ for some compact Hausdorff space Y. The function $f \in C(Y)$ corresponding to x is injective since it generates $C(Y)$; thus f is a homeomorphism from Y onto a compact subspace X of $\mathbb{C}$. Then $\sigma(f) = \sigma(x)$ (by II.1.6.7, the spectrum of x is the same whether computed in A or in $C^*(x,1)$). On the other hand, the spectrum of a function in $C(Y)$ is exactly its range, so $\sigma(f) = X$, i.e. $Y \cong \sigma(x)$. Under the correspondence, x (or f) becomes the function $f(\lambda) = \lambda$.

p. 64 II.3.1.4: The first sentence, while technically correct, might be slightly misleading. In fact, Gelfand, Naimark, and Segal were all convinced that the additional axiom was superfluous, and said so in their papers.

p. 68 II.3.2.5: In the proof, it would be better to use α_n instead of k_n.

p. 69: The notion of well-supported element can be used to give a simple proof of the following (cf. [**?**]):

Theorem. Let A be a C*-algebra. The following are equivalent:

(i) Every commutative C*-subalgebra of A is finite-dimensional.

(ii) Every positive element of A has finite spectrum.

(iii) A does not have an infinite sequence of mutually orthogonal nonzero positive elements.

(iv) Every element of A is well-supported (i.e. A is a von Neumann regular ring).

(v) Every hereditary C*-subalgebra of A has a unit.

(vi) A is reflexive as a Banach space.

(vii) A is finite-dimensional.

In particular, every infinite-dimensional C*-algebra contains C*-subalgebras which are nonunital.

In fact, no infinite-dimensional Banach algebra is a regular ring [**?**].

Proof: (vii) trivially implies the other conditions, and (i) $\Longrightarrow$ (ii) is also trivial. For (ii) $\Longrightarrow$ (iii), suppose (a_n) is a sequence of mutually orthogonal positive elements of norm 1. Then $a = \sum 2^{-n} a_n$ has infinite spectrum since $2^{-n} \in \sigma(a)$ for all n. The same proof shows (iv) $\Longrightarrow$ (iii), since a is not well-supported. (ii) $\Longrightarrow$ (iv) is obvious. (iii) $\Longrightarrow$ (i) is obvious from the fact that any infinite Hausdorff space has an infinite sequence of mutually disjoint nonempty open sets. For (v) $\Longrightarrow$ (iv), let $x \in A$. If $\overline{x^*Ax}$ has a unit p, then it follows from II.4.2.6 that x is well-supported. For a partial strong converse, if (iv) holds and B is a σ-unital C*-subalgebra (not necessarily hereditary) of A, and h is a strictly positive element of B, then h is well-supported and its support projection is a unit for B. In particular, if A itself is σ-unital, it is unital.

We now show (iv) $\Longrightarrow$ (vii). Suppose (iv) holds, and first suppose that A is separable. By the last paragraph, A is unital. Observe that every nonzero hereditary C*-subalgebra B contains a minimal projection. For if not, B contains a nonzero projection p (e.g. its unit), and it can be subdivided into p_1 and $p - p_1$, and $p - p_1$ can be subdivided into p_2 and $p - p_1 - p_2$, etc., generating a sequence of mutually orthogonal projections which contradicts (iii). So if $\{p_i\}$ is a maximal family of mutually orthogonal minimal projections in A, then $\{p_i\} = \{p_1, \dots, p_n\}$ is finite by (iii), and $p_1 + \cdots + p_n = 1$. If p_iAp_i is more than one-dimensional, there is a self-adjoint element in p_iAp_i which is not a multiple of p_i, hence a self-adjoint element with two distinct nonzero numbers in its spectrum, and so by functional calculus and (ii) two orthogonal nonzero projections in p_iAp_i, contradicting minimality of p_i. So p_iAp_i is one-dimensional. If $p_iAp_j \neq \{0\}$, let u be a unit vector in p_iAp_j; then u is a partial isometry from p_j to p_i. If $v \in p_iAp_j$, then $vu^* = \lambda p_i = (\lambda u)u^*$ for some λ, and hence $v = vp_j = vu^*u = (\lambda u)u^*u = \lambda u$. So p_iAp_j is one-dimensional. Thus A, which is spanned by $\{p_iAp_j : 1 \leq i, j \leq n\}$, is finite-dimensional.

For the general case of (iv) $\Longrightarrow$ (vii), if A satisfies (iv), then any separable C*-subalgebra of A also satisfies (iv) and is thus finite-dimensional. If A is not finite-dimensional, then A contains a sequence (x_n) of linearly independent elements. The C*-subalgebra generated by the sequence (x_n) is thus a separable infinite-dimensional C*-subalgebra of A, a contradiction.

For (vi) $\Longrightarrow$ (v), if A is reflexive as a Banach space, then A has weakly (=weak-*) compact unit ball, which has extreme points by the Krein-Milman theorem, so A must be unital by II.3.2.17. Since any closed subspace of a reflexive Banach space is reflexive, any C*-subalgebra of A is also reflexive and hence unital.

In [?], the following conditions are also shown to be equivalent to these:

(viii) A is weakly complete [a nonreflexive Banach space X cannot be weakly complete, since X is weak-* dense in X^{**} and the restriction of the weak-* topology on X^{**} to X is the weak topology. This is related in spirit to I.3.2.2, but not the same since it involves the weak topology and not the weak operator topology. Actually, an infinite-dimensional reflexive Banach space is not weakly complete either; the proof is a slight modification of the Hilbert space argument in [Hal67, Problem 30], which is similar to the argument of I.3.2.2.]

(ix) A^{**} is separable (III.5.2.7 and the addendum to p. 221).

It may seem that this theorem, at least (i) $\Longrightarrow$ (vii), should be a triviality, but it is not (although it is not a deep or difficult result). In fact, it might seem that there should be a general principle that a "large" C*-algebra should have "large" commutative C*-subalgebras. But at least some versions of this "principle" are false. For example, if G is a free group on uncountably many generators, then $C_r^*(G)$ is nonseparable, but every commutative C*-subalgebra is separable [?, 6.7] (the first such example is in [?]), and the von Neumann algebra $\mathfrak{L}(G)$ generated by the left regular representation of G is a II_1 factor which does not have separable predual, but every masa has separable predual [?, 6.4].

Actually, almost all infinite-dimensional C*-algebras contain a positive element whose spectrum is the entire interval $[0,1]$: the ones which do not are C*-algebras in which the spectrum of every self-adjoint element is countable. Such C*-algebras are called *scattered* and are Type I and AF; separable scattered C*-algebras are precisely the C*-algebras with separable dual [?] (see also the *Math Review* of this paper by S. Sakai), [?], [?], [?], [?].

This theorem should be inserted after II.4.2.6.

p. 70: The proof of II.3.2.12 shows a little more: the unitaries $a \pm i(1-a^2)^{1/2}$ have spectrum contained in the upper half circle and lower half circle respectively. Thus they are not only in $\mathcal{U}(A)_{\mathrm{o}}$, but are exponentials in

$$\mathcal{U}(A)_e = \{e^{ih} : h = h^*, \|h\| \leq \pi\}\,.$$

In the proof of II.3.2.13, v is also in $\mathcal{U}(A)_e$, since it is of distance < 2 from 1. However, it cannot be concluded from this argument that $w_1, w_2 \in \mathcal{U}(A)_e$ even if $u \in \mathcal{U}(A)_e$ since $\mathcal{U}(A)_e$ is not closed under multiplication in general (in fact, the subgroup of $\mathcal{U}(A)$ generated by $\mathcal{U}(A)_e$ is $\mathcal{U}(A)_{\mathrm{o}}$, cf. II.1.5.4). Thus the proof of II.3.2.14 does not show that the closed convex hull of $\mathcal{U}(A)_e$ is the entire closed unit ball.

The following unpublished argument of E. Kirchberg shows that if A is a unital C*-algebra, then the closed convex hull of $\mathcal{U}(A)_e$ is the entire closed unit ball. One can also use

$$\mathcal{U}(A)_{oo} = \{u \in \mathcal{U}(A) : \|1 - u\| < 2\}$$

which is a norm-dense subset of $\mathcal{U}(A)_e$ [$e^{ih} = \lim_{\epsilon\to 0} e^{i(1-\epsilon)h}$].

If M is a von Neumann algebra, then $\mathcal{U}(M)_e = \mathcal{U}(M)_o = \mathcal{U}(M)$. Thus the norm-closed convex hull of $\mathcal{U}(A^{**})_e$ is the closed unit ball of A^{**}. It follows from the Kaplansky Density Theorem and strong continuity of the exponential function that $\mathcal{U}(A)_e$ is strongly dense in $\mathcal{U}(A^{**})_e$, and hence the σ-weakly closed convex hull of $\mathcal{U}(A)_e$ in A^{**} is the closed unit ball of A^{**}. So if $\phi \in A^*$ satisfies $Re\,\phi(u) \leq 1$ for all $u \in \mathcal{U}(A)_e$, its σ-weakly continuous extension ϕ^{**} to A^{**} satisfies $Re\,\phi^{**}(x) \leq 1$ for all $x \in A^{**}$, $\|x\| \leq 1$, i.e. $Re\,\phi(x) \leq 1$ for all $x \in A$, $\|x\| \leq 1$. So the norm-closed convex hull of $\mathcal{U}(A)_e$ in A is the closed unit ball by the Hahn-Banach Theorem.

It is not known whether the u_j in the conclusion of II.3.2.14 can be chosen in $\mathcal{U}(A)_e$, or even whether the convex hull of $\mathcal{U}(A)_e$ contains the open unit ball of A. It is shown in [Haa90] that if $\|x\| \leq \frac{1}{3}$, then x can be written as an average of three elements of $\mathcal{U}(A)_e$. A simple analog of II.3.1.2 shows that any element of norm $\leq \frac{1}{2}$ is an average of four exponentials of the form e^{ih}, but without the norm restriction on h.

It is interesting to consider the subset of $\mathcal{U}(A)_o$ consisting of all exponentials e^{ih}, $h = h^*$, with no norm restriction on h. If A is commutative, this set is exactly $\mathcal{U}(A)_o$, but this usually fails for noncommutative A. For many A, this set of exponentials is dense in $\mathcal{U}(A)_o$; such C*-algebras are said to have *exponential rank* $1 + \epsilon$. See [**?**] for a discussion of exponential rank and its relation to the structure of C*-algebras.

p. 73 II.3.3.1: (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) are essentially trivial, but (ii) $\Longrightarrow$ (iii) is not immediately obvious. This implication is a special case of the following proposition (note that $p \leq \lambda q$ for some $\lambda > 0$ implies $\lambda^{-1}p \leq q$):

Proposition. Let A be a C*-algebra, $a \in A_+$, q a projection in A. If $a \leq q$, then $aq = qa = a$.

Proof: We have $0 \leq (1-q)a(1-q) \leq (1-q)q(1-q) = 0$ by II.3.1.8. Hence, if $x = a^{1/2}(1-q)$, we have $x^*x = 0$, so $x = 0$, $a^{1/2} = a^{1/2}q$, $a = a^{1/2}(a^{1/2}q) = aq$, $a = a^* = q^*a^* = qa$.

p. 74 II.3.3.5: Here is a more general result with a somewhat simpler proof which gives an explicit formula for δ [**?**]:

Proposition. Let A be a C*-algebra, p, q projections in A, and $0 < \delta \leq \frac{1}{4}$. If $\|pq - qp\| < \delta$, then there is a projection $p' \in A$ with $\|p - p'\| < 3\delta$ and $p'q = qp'$.

We need a lemma which should be inserted as Proposition II.3.1.14 on p. 67:

Lemma. Let A be a C*-algebra, $x, y \in A$, p a projection in A. Then

$$\|px(1-p)+(1-p)yp\| = \max(\|px(1-p)\|, \|(1-p)yp\|)\,.$$

In particular, if $x = x^*$, then

$$\|x - [pxp + (1-p)x(1-p)\| = \|px(1-p)+(1-p)xp\| = \|(1-p)xp\|\,.$$

Proof: Let $z = px(1-p)+(1-p)yp$. Then

$$\|z\|^2 = \|z^*z\| = \|(1-p)x^*px(1-p)+py^*(1-p)yp\|$$

$= \max(\|(1-p)x^*px(1-p)\|, \|py^*(1-p)yp\|) = \max(\|px(1-p)\|^2, \|(1-p)yp\|^2)$ since $(1-p)x^*px(1-p) \perp py^*(1-p)yp$.

To prove II.3.3.5 from the Proposition, set $\delta = \min\left(\frac{1}{4}, \frac{\epsilon}{3}\right)$. If $\|pq - q\| < \delta$, then

$$\|pq - qp\| = \|pq(1-p) - (1-p)qp\| = \|pq(1-p)\| = \|p(q-qp)\| < \delta$$

so there is a p' with $\|p - p'\| < 3\delta \leq \epsilon$ and $p'q = qp'$. Then $p'q$ is a projection $\leq q$, and $\|p'q - q\| \leq \|p'q - pq\| + \|pq - q\| < 4\delta \leq 1$, so $p'q = q$.

We now prove the Proposition. Let $\alpha = \|qp - pq\|$. We have

$$\|p - [qpq + (1-q)p(1-q)]\| = \|(1-q)pq\| = \|(pq-qp)q\| \leq \|qp - pq\| = \alpha\,.$$

Also,

$$\|qpq - (qpq)^2\| = \|qpq - qpqpq\| = \|qp(pq-qp)q\| \leq \alpha$$

and so $\sigma(qpq) \subseteq [0,\gamma] \cup [1-\gamma, 1]$, where $\gamma = \frac{1-\sqrt{1-4\alpha}}{2} < 2\alpha < \frac{1}{2}$ since $\alpha < \frac{1}{4}$. Thus by functional calculus there is a projection $r \in qAq$ with $\|r - qpq\| \leq \gamma < 2\delta$. Similarly, there is a projection $s \in (1-q)A(1-q)$ with $\|s-(1-q)p(1-q)\| < 2\delta$. If $p' = r + s$, then

$$\|p' - p\| \leq \|p' - [qpq + (1-q)p(1-q)]\| + \|[qpq + (1-q)p(1-q)] - p\| < 3\delta\,.$$

p. 76 II.3.4.3(iii): In the literature on the Cuntz semigroup, what we call $\approx$ is often denoted $\sim$.

p. 79 II.3.2.14 succeeding comment: "… λv cannot be written as a convex combination of n unitaries in any containing C*-algebra (in which v is an isometry, i.e. with the same unit as $C^*(v)$) [KP85]."

p. 81 II.4.2.3: last line should be $\phi(h) > 0$.

There was no proof given of the last statement; the proof is a bit tricky and should appear somewhere in II.6. If $h \in A_+$ and $\phi \in \mathcal{S}(A)$ with $\phi(h) = 0$, then

ϕ vanishes on $h^{1/2}A_+h^{1/2}$ since if $a \in A_+$, then $0 \leq h^{1/2}ah^{1/2} \leq \|a\|h$, and hence ϕ vanishes on $h^{1/2}Ah^{1/2}$ and thus on hAh; so h is not strictly positive.

The following slick proof of the converse is adapted from [Ped79,3.10.5]. If A is a C*-algebra, let $\mathcal{F}(A)$ be the set of positive linear functionals on A of norm ≤ 1. $\mathcal{F}(A)$ is a compact convex set. There is a positive map from A to $C(\mathcal{F}(A))$ sending y to $\hat{y}$, where $\hat{y}(\phi) = \phi(y)$; this map is isometric by II.6.3.3.

Let $h \in A_+$ with $\phi(h) > 0$ for all $\phi \in \mathcal{S}(A)$; we may assume $\|h\| = 1$. Then, for any nondegenerate representation $\pi : A \to \mathcal{L}(\mathcal{H})$ and nonzero vector $\xi \in \mathcal{H}$, the map $\psi(x) = \langle \pi(x)\xi, \xi\rangle$ is a nonzero positive linear functional on A, and thus $\langle \pi(h)\xi, \xi\rangle > 0$. So the support projection of $\pi(h)$ in $\mathcal{L}(\mathcal{H})$ is I. The bounded increasing sequence $(\pi(h^{1/n}))$ converges strongly to the support projection of $\pi(h)$, i.e. $\pi(h^{1/n}) \to I$ strongly. In particular, if $x \in A$ is fixed, and $\phi \in \mathcal{F}(A)$, applying this to π_ϕ we get $\phi(z_n) \searrow 0$, where $z_n = x^*(1 - h^{1/n})^2 x$. Thus $\hat{z}_n$ decreases to 0 pointwise, hence uniformly by Dini's Theorem, and so $\|(1 - h^{1/n})x\| \to 0$, $h^{1/n}x \to x$. This is true for all $x \in A$, so $(h^{1/n})$ is an approximate unit for A.

p. 87 last line: $I_a = \left\{ \sum_{j=1}^{n} y_j a z_j \quad : \quad n \in \mathbb{N},\ y_j, z_j \in A \right\}$

p. 92 last line of II.5.3.12: "... generated by h_λ ..."

p. 95 II.5.5.1 Add at end: See also the addendum to p. 137.

p. 105 II.6.2.7: I don't know what I was thinking when I wrote the last sentence, but $\mathcal{S}(A)$ is never locally compact if A is nonunital. Let $\mathcal{F}(A)$ be the set of positive linear functionals on A of norm ≤ 1; $\mathcal{F}(A)$ is a compact convex set, and $\mathcal{S}(A)$ is a face in $\mathcal{F}(A)$, which is closed if A is unital. The result for nonunital A follows from the next proposition and the fact that $\mathcal{S}(A)$ is not open in $\mathcal{F}(A)$ [if $\phi \in \mathcal{S}(A)$, then $\phi = \lim_{\epsilon \to 0}(1 - \epsilon)\phi$.]

Proposition. If A is nonunital, then $\mathcal{S}(A)$ is dense in $\mathcal{F}(A)$.

Proof: First note that if A is nonunital, then A has an approximate unit (h_λ) for which there there are states ψ_λ with $\psi_\lambda(h_\lambda) = 0$. One only needs an approximate unit in which the h_λ are not strictly positive. If A is not σ-unital, any approximate unit will do; if A is σ-unital, choose an almost idempotent approximate unit. The net (ψ_λ) converges weak-* to 0 (note that ψ_λ vanishes on $h_\lambda^{1/2}Ah_\lambda^{1/2}$ and hence on $h_\lambda A h_\lambda$). If $\phi \in \mathcal{F}(A)$ with $\|\phi\| = \alpha$, set $\phi_\lambda = \phi + (1 - \alpha)\psi_\lambda$. Then $\phi_\lambda \in \mathcal{S}(A)$ and $\phi_\lambda \to \phi$.

p. 112 line -4: The reference should be to II.2.2.10 in the addendum to p. 61. [The problem with just using II.1.6.5 is that it is not obvious that A/J is complete with respect to the norm given by the formula.]

p. 113 II.6.5.7: This is stated sloppily, since there are several inequivalent definitions of local compactness used for non-Hausdorff spaces. If A is a C*-algebra, then II.6.5.6 shows that every point of Prim(A) has a neighborhood base of compact sets. But a point need not have any closed compact neighborhood.

As an example where this fails, let $(\mathcal{H}_n)$ be a sequence of separable infinite-dimensional Hilbert spaces, with unit vectors ξ_n. Set $\mathcal{H} = \bigotimes_{n=1}^{\infty}(\mathcal{H}_n, \xi_n)$, $\mathcal{H}'_m = \bigotimes_{n=1}^{m} \mathcal{H}_n$, $\mathcal{H}''_m = \bigotimes_{n=m+1}^{\infty}(\mathcal{H}_n, \xi_n)$. Then $\mathcal{H} \cong \mathcal{H}'_m \otimes \mathcal{H}''_m$ for each m. Let A be the C*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $\mathcal{K}(\mathcal{H})$ and $1 \otimes \mathcal{K}(\mathcal{H}''_m)$ for each m. A is a separable type I C*-algebra, which is scattered (p. 69 addendum) and AF, and the ideals of A form an increasing sequence. Prim(A) is a sequence of points, and the closure of each point is the point along with all points above it. No point of Prim(A) has a closed compact neighborhood.

p. 130 II.6.9.18: There is also an approximate version [Cho74]:

Proposition. Let $\phi : A \to B$ be a completely positive contraction, and $x, y \in A$. Then

$$\|\phi(yx) - \phi(y)\phi(x)\|^2 \leq \|\phi(yy^*) - \phi(y)\phi(y)^*\| \|\phi(x^*x) - \phi(x)^*\phi(x)\| .$$

p. 130 Add at bottom of page:

The adjoint operation on a C*-algebra is also completely determined by the algebra structure and the norm (see the addendum to p. 137).

p. 137: At the end of Section II.6, add:

Uniqueness of the Adjoint

Proposition. A norm-decreasing complex-linear homomorphism from a C*-algebra A to a C*-algebra B is a *-homomorphism. In particular, if two C*-algebras are isometrically isomorphic as complex algebras, they are *-isomorphic.

Proof: If $\phi : A \to B$ is a norm-decreasing complex-linear homomorphism, then $\phi^{**} : A^{**} \to B^{**}$ is also a norm-decreasing complex-linear homomorphism. Thus we may assume that A is unital. Then $p = \phi(1)$ is an idempotent in B of norm 1, hence a projection, and the range of ϕ is contained in pBp. Replacing B by pBp, we may assume that ϕ is unital. Then ϕ is positive by II.6.9.4, and in particular *-preserving by II.1.5.9 and II.3.1.2(vi).

In fact, a norm-decreasing real-linear homomorphism from one C*-algebra to another is *-preserving [**?**, 4.5.5,5.6] (but not necessarily complex-linear). But two C*-algebras which are isometrically isomorphic as real Banach algebras, hence *-isomorphic as real C*-algebras, are not necessarily isomorphic as (complex) C*-algebras: the map $x \mapsto x^*$ is an isometric real-linear *-isomorphism from A to A^{op}, but A and A^{op} need not be isomorphic as C*-algebras (IV.1.7.16).

If two C*-algebras are isomorphic as complex algebras, are they *-isomorphic as complex algebras, and hence isometrically isomorphic as C*-algebras?

Corollary. Let A be a complex Banach algebra. Then there is at most one involution on A making A a C*-algebra with the given multiplication and norm.

Proof: If $x \mapsto x^*$ and $x \mapsto x^\star$ are involutions on A making A into a C*-algebra, apply the Proposition to the identity map from $(A, *)$ to $(A, \star)$.

How does one actually derive the adjoint from the multiplication and norm?

p. 137 I.7.1.1: Add (in two places) that $\mathcal{E}$ is not only a right B-module but also a complex vector space, and that the scalar multiplication on $\mathcal{E}$ is compatible with the scalar multiplication on B in the sense that $\lambda(\xi b) = \xi(\lambda b)$ for all $\xi \in \mathcal{E}, \lambda \in \mathbb{C}, b \in B$ (it then follows that $(\lambda\xi)b = \lambda(\xi b)$ for all λ, ξ, b). [That $\mathcal{E}$ is a complex vector space is implicit in (i), but the compatibility of scalar multiplication is vital and not automatic.]

p. 141 II.7.2.1: I have never liked the term *adjointable*. Some colleagues recommended that I not use it, but I perhaps unwisely did not follow this advice, because I didn't have a satisfactory alternative. In [**?**] (and perhaps in other references too) the term *jointed* is used instead. Frankly, I'm not sure it is any better.

p. 142 first paragraph: The fact that $\mathcal{L}(\mathcal{E})$ is a C*-algebra is important enough that it should be explicitly stated as a proposition. For the proof, refer to I.2.3.1.

p. 142 II.7.2.3: add at end:

We have $\|T\| = \sup \|T_\psi\|$, where ψ runs over all states of B.

p. 143 II.7.2.7: There are no misstatements, but the paragraph mixes apples and oranges. It should be stylistically broken in two: the sentence "Even an isometric ..." should begin a new paragraph, and the word "Even" deleted.

p. 145 II.7.3.3: Proof should begin new line. (This was correct in the TeX file I submitted, and the error was introduced after I reviewed the proof sheets!)

p. 145 Proof of II.7.3.7: "... Then L is a left centralizer of $\mathcal{K}(\mathcal{H})$, ..."

p. 147 II.7.3.12(iv): The multiplier algebra of $C_o(X, B)$ is isomorphic to the C*-algebra of (norm-)bounded strictly continuous functions from X to $M(B)$. Not every such function extends to a strictly continuous function from βX to $M(B)$ in general ([APT73,3.8], [GJ76,6N]).

p. 153 II.7.6.11: It is worth stating the result proved in [Bro77]:

Theorem. Let A be a C*-algebra, and B a full hereditary C*-subalgebra of A. If A and B are σ-unital, then the inclusion map from B into A extends to an isomorphism from $B \otimes \mathbb{K}$ onto $A \otimes \mathbb{K}$.

The case where A and B are both unital (i.e. B is a full corner in A) is technically simpler to prove (the key step is V.2.4.10); see [**?**].

p. 155 II.8.1.4: There is a subtlety in the proof which is not explained: in order to make sense of $f(x)$ (and $f(\pi(x))$), we must have $\sigma(x) \subseteq X$ (and $\sigma(\pi(x)) \subseteq X$, but $\sigma(\pi(x)) \subseteq \sigma(x)$). This follows from

Proposition. Let $\{A_i : i \in I\}$ be C*-algebras, and $A = \prod A_i$. If $x = (\cdots x_i \cdots)$ is a normal element of A, then $\sigma_A(x) = \overline{\cup_i \sigma_{A_i}(x_i)}$.

Proof: If not all the A_i are unital, then $0 \in \sigma_A(x)$ and $0 \in \sigma_{A_i}(x_i)$ for some i. Thus, since $\prod A_i \subseteq \prod \tilde{A}_i$, by II.1.6.7 we may assume each A_i is unital. Then $y = (\cdots y_i \cdots) \in A$ is invertible if and only if each y_i is invertible and $\{\|y_i^{-1}\| : i \in I\}$ is bounded. Thus $\sigma_{A_i}(x_i) \subseteq \sigma_A(x)$ for all i, so $X := \overline{\cup_i \sigma_{A_i}(x_i)} \subseteq \sigma_A(x)$. If $\lambda \in \mathbb{C} \setminus X$, then $x_i - \lambda 1$ is invertible in A_i for each i and $\|(x_i - \lambda 1)^{-1}\| \leq d^{-1}$, where d is the distance from λ to X, by I.2.6.17 above, so

$$(x - \lambda 1)^{-1} = (\cdots (x_i - \lambda 1)^{-1} \cdots) \in A.$$

If x is not normal, then $\overline{\cup_i \sigma_{A_i}(x_i)} \subseteq \sigma_A(x)$, but equality does not hold in general. For example, let T_n be as in I.2.6.17 above, and in $A = \prod_{\mathbb{N}} \mathcal{L}(l^2)$ let $T = (\cdots T_n \cdots)$. Then $\sigma(T_n) = \{1\}$ for all n, but T is not invertible in A, so $0 \in \sigma(T)$. (In fact, it is not hard to see that $\sigma(T)$ is the closed disk of radius 1 around 1.)

p. 158 II.8.3.1: There is a question whether polynomials with constant term should be allowed as relations in the general construction of universal C*-algebras (as opposed to universal unital C*-algebras). There is no harm in allowing polynomials with constant term; in a representation, such a constant term is interpreted as the corresponding multiple of the identity.

The only subtlety is that a polynomial with constant term may not be realizable among bounded operators. The most famous example is the Canonical Anticommutation Relation from quantum mechanics $p(x, y) = xy - yx - 1$. [Suppose x and y satisfy this relation. If $\lambda \in \sigma(xy)$, $\lambda \neq 0$, then $\lambda \in \sigma(yx)$ by II.1.4.2(iv), so $\lambda + 1 \in \sigma(yx+1) = \sigma(xy)$; if $\lambda \in \sigma(xy)$, $\lambda \neq 1$, then $\lambda \in \sigma(yx+1)$, so $\lambda - 1 \in \sigma(yx)$, hence $\lambda - 1 \in \sigma(xy)$. Thus $\sigma(xy)$ is unbounded, a contradiction.] Thus, for a set $(\mathcal{G}|\mathcal{R})$ where the relations contain constant terms, it must be verified that there is at least one representation in order for $C^*(\mathcal{G}|\mathcal{R})$ to be defined. (If the relations do not contain constant terms, there is always at least one representation, setting all generators equal to 0.)

p. 161 lines -11-12: "… and the first three relations imply that $\alpha^*\alpha = \alpha\alpha^*$, …"

p. 182 II.9.2.6: Note, however, that a mixture of $\otimes_{\max}$ and $\otimes_{\min}$ is not associative. The map $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$ extends to a homomorphism from $(A \otimes_{\min} B) \otimes_{\max} C$ to $A \otimes_{\min} (B \otimes_{\max} C)$, defined as follows. Let π and ρ be faithful representations of A and $B \otimes_{\max} C$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $\pi \otimes \rho$ is a faithful representation of $A \otimes_{\min} (B \otimes_{\max} C)$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then ρ

defines commuting representations ρ_B and ρ_C of B and C on $\mathcal{H}_2$, and $\pi \otimes \rho_B$ and $1 \otimes \rho_C$ give commuting representations of $A \otimes_{\min} B$ and C on $\mathcal{H}_1 \otimes \mathcal{H}_2$, giving a representation of $(A \otimes_{\min} B) \otimes_{\max} C$ agreeing with $\pi \otimes \rho$ on $A \odot B \odot C$.

But this map is not injective in general. For example, if B is unital, the C*-subalgebra $(A \otimes 1) \otimes_{\max} C$ of $(A \otimes_{\min} B) \otimes_{\max} C$ is isomorphic to $A \otimes_{\max} C$ [if π and ρ are faithful representations of $A \otimes_{\max} C$ and B on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the faithful representation $\pi \otimes 1$ of $A \otimes_{\max} C \cong (A \otimes 1) \otimes_{\max} C$ extends to the representation $\pi \otimes \rho$ of $(A \otimes_{\min} B) \otimes_{\max} C$], but the C*-subalgebra $A \otimes_{\min} (1 \otimes C)$ of $A \otimes_{\min} (B \otimes_{\max} C)$ is isomorphic to $A \otimes_{\min} C$.

See IV.3.1.1 for a closely related result.

p. 189-190 II.9.6.8-II.9.6.9: I was rather careless in including the last paragraph of II.9.6.9 (beginning with "If C is simple and exact, . . . "), which was the result of an informal discussion shortly before the publication deadline. In fact, it is still *not* known whether the intersection formula holds in general, and thus the statement in II.9.6.8 is at least premature if not erroneous.

E. Kirchberg has provided the following corrected discussion of the situation:

If A and D are C*-algebras and C is a C*-subalgebra of D, define

$$F(A, C; A \otimes D) = \{x \in A \otimes D | R_\phi(x^* x) \in C \, \forall \phi \in \mathcal{S}(A)\}$$

(all tensor products are minimal tensor products), where R_ϕ is the right slice map of ϕ (II.9.7.1).

A C*-algebra A has the *slice map property*, or *Property (S)* (of S. Wassermann), if for every C*-algebra D and C*-subalgebra C of D, $F(A, C; A \otimes D) = A \otimes C$.

One can show:

1. Every C*-algebra with Property (S) is exact (IV.3.4.2).

2. A has Property (S) if and only if A is an inductive limit of a directed set $\{A_i\}$ of separable C*-subalgebras with Property (S).

3. A separable C*-algebra A has Property (S) if and only if A is exact and, for some embedding $A \subseteq O_2$,

$$(O_2 \otimes C) \cap (A \otimes D) = A \otimes C$$

   for all C*-algebras D and C*-subalgebras C of D.

4. The question whether Property (S) is equivalent to exactness reduces to the question of whether

$$(O_2 \otimes B) \cap (A \otimes \mathcal{L}(\mathcal{H})) = A \otimes B$$

   for each simple $A \subseteq O_2$ and von Neumann algebra $B \subseteq \mathcal{L}(\mathcal{H})$.

5. The question whether Property (S) is equivalent to exactness is also equivalent to the following question:

   Let u be a unitary in the CAR algebra A and B any unital C*-algebra. Is

$$\{u \otimes 1\}' \cap (A \otimes B) = (\{u\}' \cap A) \otimes B\,?$$

p. 193 End of introduction add: A good comprehensive reference for crossed products is [**?**].

Add this comment also on p. 212 II.10.6.

p. 197 add at bottom of page:

The same argument shows that if B is a C*-algebra, $\mathcal{E}$ is a Hilbert B-module, and π is a strictly continuous homomorphism from G to $\mathcal{U}(\mathcal{L}(\mathcal{E}))$, then π induces an integrated form nondegenerate *-homomorphism from $L^1(G)$ to $\mathcal{L}(\mathcal{E})$.

p. 198 II.10.2.2: "Theorem. Every nondegenerate representation of $L^1(G)\ldots$"

Insert before paragraph beginning "One could try to use ...":

The same arguments show more generally that if B is a C*-algebra and $\mathcal{E}$ is a Hilbert B-module, and π is a nondegenerate *-homomorphism from $L^1(G)$ into $\mathcal{L}(\mathcal{E})$, then there is a strictly continuous homomorphism from G to the unitary group of $\mathcal{L}(\mathcal{E})$ giving π as its integrated form.

Add before the paragraph in II.10.2.3 beginning "It is easy to see ...":

Using II.7.2.3, it follows that $\|f\| = \sup\{\|\pi(f)\|\}$, where π runs over all nondegenerate *-homomorphisms from $L^1(G)$ to $\mathcal{L}(\mathcal{E})$ for all Hilbert modules $\mathcal{E}$.

Add at end of II.10.2.4:

As before, there is also a one-one correspondence between strictly continuous unitary representations of G on a Hilbert B-module $\mathcal{E}$ and nondegenerate *-homomorphisms from $L^1(G)$ to $\mathcal{L}(\mathcal{E})$, for any $\mathcal{E}$ over any B.

p. 198 last two lines: The fact that the quotient map from $C^*(G)$ to $C^*_r(G)$ is an isomorphism if and only if G is amenable is worth stating as a theorem; it should go as II.10.2.12 on p. 199 after the II.10.2.11 addendum. For the proof, refer to II.10.1.9(v), [Ped79,7.3.9], and [**?**, A.18].

p. 199 II.10.2.6: See [**?**], [**?**], and [**?**] for more results on groups with simple reduced C*-algebras.

p. 199 Add after II.10.2.10:

Functoriality

II.10.2.11 Functoriality of the full or reduced C*-algebra construction is rather subtle: if $\phi : H \to G$ is a continuous group homomorphism, there is not in general an induced homomorphism from $C^*(H)$ to $C^*(G)$, or from $C^*_r(H)$ to $C^*_r(G)$. For example, consider the inclusion of $\mathbb{Z}$ into $\mathbb{R}$.

In general, ϕ defines a strictly continuous homomorphism from H to the unitary group of the multiplier algebra $M(C^*(G))$, which has an integrated form (cf. the addendum to p. 197) giving a *-homomorphism from $C^*(H)$ to $M(C^*(G))$. The same process gives a *-homomorphism from $C^*_r(H)$ to $M(C^*_r(G))$.

If H is a closed subgroup of G, these maps are injective. Injectivity is not obvious, since a unitary representation of H does not obviously extend to a unitary representation of G, but there is a process called *induction*, which converts a representation π of H into a representation $\mathrm{Ind}_H^G \pi$ of G on a different Hilbert space, and $(\mathrm{Ind}_H^G \pi)|_H$ is equivalent to a multiple of π. The idea of the construction is simple, but the details are somewhat complicated, and are omitted here; see e.g. [**?**, p. 153]. The same construction shows the map from $C^*_r(H)$ to $M(C^*_r(G))$ is injective, since if λ_H is the left regular representation of H, then $\mathrm{Ind}_H^G \lambda_H$ is equivalent to λ_G.

If G is abelian and H is a closed subgroup of G, the map can be described explicitly, since $\hat{H} \cong \hat{G}/H^\perp$, and $C^*(H) \cong C_0(\hat{G}/H^\perp)$ can be regarded as bounded continuous functions on $\hat{G}$ constant on cosets of $H^\perp$; these act naturally as multipliers on $C^*(G) \cong C_0(\hat{G})$.

In some cases $C^*(H)$ maps into $C^*(G)$:

If H is an open subgroup of G, then $C_c(H)$ may be regarded as the *-subalgebra of $C_c(G)$ of functions supported on H, which extends to an embedding of $C^*(H)$ into $C^*(G)$ agreeing with the map above.

If H is an abelian group, K a closed subgroup, and ϕ is the quotient map from H to $G = H/K$, then $\hat{G}$ is the closed subgroup $K^\perp$ of $\hat{H}$, and the induced map is just the restriction map from $C^*(H) \cong C_0(\hat{H})$ to $C^*(G) \cong C_0(K^\perp)$. Thus $C^*(G)$ is a quotient of $C^*(H)$.

It is true in general that if G is a quotient of H, then $C^*(G)$ is a quotient of $C^*(H)$, since every unitary representation of G can be regarded as a unitary representation of H.

p. 201 II.10.3.6: add before paragraph beginning "We may also put a norm . . . ":

As in the group case, if B is a C*-algebra and $\mathcal{E}$ is a Hilbert B-module, and $\pi : A \to \mathcal{L}(E)$ is a nondegenerate *-homomorphism from A to $\mathcal{L}(\mathcal{E})$, and ρ a strongly continuous unitary representation of G on $\mathcal{E}$, satisfying the covariance condition, there is an integrated form nondegenerate *-homomorphism $\pi \rtimes \rho$ of $L^1(G, A)$ to $\mathcal{L}(\mathcal{E})$, and every such *-homomorphism arises in this way.

Add after last line giving the formula for $\|f\|$:

By II.7.2.3, $\|f\| = \sup\{\|(\pi \rtimes \rho)(f)\|\}$, where (π, ρ) range over covariant pairs of representations on arbitrary Hilbert modules. See [**?**, 2.40] for details.

p. 203 II.10.3.13: Formula for π_α should be $([\pi_\alpha(x)]\xi)(t) = \pi(\alpha_t(x))(\xi(t))$.

p. 203 Comment after II.10.3.14: This should be isolated as a theorem, on p. 205 as II.10.3.22 after the II.10.3.21 addendum. For the proof, refer to [**?**, 7.13].

p. 205 Add after II.10.3.20 (this should more logically go after II.10.3.15 on p. 204):

Functoriality

II.10.3.21 Both the full and reduced crossed products are functorial in A and sometimes in G. The functoriality in A is straightforward: if α and β are continuous actions of G on C*-algebras A and B, and $\phi : A \to B$ is an equivariant *-homorphism (i.e. $\beta_g \circ \phi = \phi \circ \alpha_g$ for all $g \in G$), then there is an induced *-homomorphism from $A \rtimes_\alpha G$ to $B \rtimes_\beta G$, defined in the obvious way using the induced *-homomorphism from $L^1(G, A)$ to $L^1(G, B)$. The same process gives an induced *-homomorphism from $A \rtimes_\alpha^r G$ to $B \rtimes_\beta^r G$.

Functoriality in G has the same subtleties as in the group C*-algebra case, leading only to a map into the multiplier algebra in most cases. If α is an action of G on A, and H is an open subgroup of G, then there are natural embeddings of $A \rtimes_\alpha H$ into $A \rtimes_\alpha G$ and from $A \rtimes_\alpha^r H$ into $A \rtimes_\alpha^r G$.

p. 213 II.10.7.6: Original reference for twisted crossed products is [**?**].

p. 215 II.10.8.12: Second paragraph "... a cocommutative finite quantum group is of the form $\mathbb{C}G$."

p. 218 II.10.8.20: Reference [**?**] for deformation quantizations omitted.

p. 221 Second paragraph: To show that an infinite-dimensional von Neumann algebra M is not norm-separable, note that M contains a sequence (p_n) of mutually orthogonal nonzero projections (see the addenda to p. 69). If $S \subseteq \mathbb{N}$, set $p_S = \sum_{n \in S} p_n$. Then $\{p_S : S \subseteq \mathbb{N}\}$ is an uncountable set of projections in M, and $\|p_S - p_{S'}\| = 1$ if $S \neq S'$. In fact, if $(\alpha_1, \alpha_2, \dots)$ is a bounded sequence of complex numbers, then $\sum \alpha_n p_n$ converges strongly in M, and $(\alpha_1, \alpha_2, \dots) \mapsto \sum \alpha_n p_n$ gives an injective *-homomorphism from l^∞ into M.

p. 243: add at the end of III.1.7:

III.1.7.12 Proposition. If J is a nontrivial norm-closed ideal in a factor M, then M/J is not countably decomposable; in fact, M/J contains uncountably many mutually orthogonal projections each equivalent to the identity.

Proof: Write $M = N \bar{\otimes} \mathcal{L}(\mathcal{H})$ as in III.1.7(iv), with $N \otimes \mathcal{K}_\beta(\mathcal{L}(\mathcal{H}))$ generating J. Then M/J contains $(1 \otimes \mathcal{L}(\mathcal{H}))/(1 \otimes \mathcal{K}_\beta(\mathcal{L}(\mathcal{H})))$. So it suffices to show the result for $M = \mathcal{L}(\mathcal{H})$, $J = \mathcal{K}_\beta(\mathcal{L}(\mathcal{H}))$. If $\mathcal{H}$ is nonseparable, then $\mathcal{L}(\mathcal{H})$ contains uncountably many mutually orthogonal projections equivalent to the identity, so every quotient has the same property.

So it suffices to show that the Calkin algebra contains an uncountable set of mutually orthogonal projections equivalent to the identity. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. There is an uncountable collection $\{S_r : r \in \mathbb{R}\}$ of infinite subsets of $\mathbb{N}$, the intersection of any two of which is finite: fix an enumeration of $\mathbb{Q}$, and for each real number r let S_r correspond to a sequence of distinct rational numbers converging to r. Let $\{e_{mn} : m, n \in \mathbb{N}\}$ be a set of matrix units in $\mathcal{L}(\mathcal{H})$, and set $p_r = \sum_{n \in S_r} e_{nn} \in \mathcal{L}(\mathcal{H})$ and q_r its image in the Calkin algebra. For each r, p_r is infinite-rank and hence equivalent to the identity in $\mathcal{L}(\mathcal{H})$; but $p_r p_s$ is finite-rank for $r \neq s$, so $q_r \perp q_s$ in the Calkin algebra.

p. 243-244 Section III.1.8: In his review of this volume in the *Mathematical Reviews*, P. Jolissaint criticized this section as "superfluous." In my opinion, this comment is simply a graphic illustration of the extreme prejudice many operator algebraists have against AW*-algebras, a prejudice I do not personally share. While I think it would be a mistake to make AW*-algebras a central focus of a work on operator algebras (and they certainly are not in this volume), and I would caution any young operator algebraist against making AW*-algebras the focus of his/her research career without very good reason, I do not think inclusion of a one-page section on AW*-algebras in a 517-page book is at all excessive. In fact, I would say that no book on operator algebra theory could claim to be comprehensive without mention of AW*-algebras. In any event, inclusion of this brief section is fully justified by II.6.8.16, III.2.4.4-III.2.4.5, and IV.2.1.7, if nothing else. In addition, it is a historical fact that in their first attempt [MVN36] to prove that a II_1 factor has a trace, Murray and von Neumann really proved that every II_1 AW*-factor has a quasitrace (cf. III.1.7.10; of course, neither the term "AW*-algebra" nor "quasitrace" had yet been coined.)

See [**?**] for more comments on the place and role of AW*-algebras in the general theory of operator algebras.

p. 243 Intro to III.1.8: delete parenthetical phrase and add new paragraph:

A C*-algebra which is isomorphic to a von Neumann algebra is called a *W*-algebra*, where the "W" stands for "weak". (Caution: in many references, the term "W*-algebra" is actually synonymous with "von Neumann algebra", i.e. a concrete weakly closed *-algebra of operators. Our definition, which has been somewhat less commonly used, is more useful and, I believe, more appropriate. One could then talk about "concrete W*-algebras", i.e. von Neumann algebras, and "abstract W*-algebras", as is done with C*-algebras.)

p. 244 II.1.8.3: add:

An AW*-algebra must be unital. It follows easily from II.1.8.2 that a commutative C*-algebra $C(X)$ is an AW*-algebra if and only if X is *extremally disconnected* or *stonean* (sometimes spelled *stonian*), i.e. the closure of every open set is open.

An infinite extremally disconnected compact Hausdorff space contains a copy of $\beta\mathbb{N}$ and thus has cardinality at least $2^{2^{\aleph_0}}$ [GJ76,9H]. Thus the only metrizable extremally disconnected compact Hausdorff spaces are the finite discrete spaces, and an infinite-dimensional commutative AW*-algebra is nonseparable. Since a masa in an AW*-algebra is a (commutative) AW*-algebra, it follows from the addendum to p. 69 that every infinite-dimensional AW*-algebra is nonseparable. (The argument in the addendum to p. 221 above also works in AW*-algebras.)

p. 244 III.1.8.6: This result is not quite correctly stated, even though the statement is essentially identical to the one in [Wri80], because the definition of

"W*-algebra" in [Wri80] is different from the one in this volume (in [Wri80], a "W*-algebra" is actually a von Neumann algebra). The correct statement using our definitions is:

Theorem. Let M be an AW*-algebra on a separable Hilbert space. If $\mathcal{Z}(M)$ is a von Neumann algebra, then M is a von Neumann algebra. In particular, any AW*-factor which can be (faithfully) represented on a separable Hilbert space is a W*-algebra.

It could well be that the statement of III.1.8.6 is true and equivalent to this statement by elementary arguments along the lines of the addenda to p. 252.

There is a similar slight ambiguity in the statement of IV.2.1.7 (p. 353).

p. 244 III.1.8.7: There are many interesting structure results about AW*-algebras analogous to the theory of W*-algebras, some of which are much harder to prove in the AW* case. For example, it is virtually trivial that a matrix algebra over a W*-algebra is a W*-algebra. It is true, although a surprisingly complicated theorem to prove, that a matrix algebra over an AW*-algebra is also an AW*-algebra ([**?**], [Ber72]). The fact that there is no topology on a general AW*-algebra analogous to the σ-weak or σ-strong topology on a W*-algebra causes considerable complications in the theory.

Variations on the theory of AW*-algebras have been considered. A *Rickart C*-algebra* ([**?**], where they are called "B_p^*-algebras") is a unital C*-algebra in which the right annihilator of every element (equivalently, of every countable set) is generated by a projection. These are precisely the unital C*-algebras in which every element has left and right support projections in the sense of I.5.2.1 or II.3.2.9. See [Ber72] and [**?**] for details of the theory of Rickart C*-algebras, which have primarily been studied by experts with a strong interest in ring theory. Generalizing AW*-algebras in another direction are the SAW*-algebras of G. Pedersen [**?**]; primary examples (besides AW*-algebras themselves) are *corona algebras*, C*-algebras of the form $M(A)/A$ for a separable nonunital C*-algebra A. Properties of these types of C*-algebras are noncommutative versions of certain phenomena occurring in compact Hausdorff spaces in the absence of first countability.

p. 249 III.2.2.3: add:

Conversely, if π is a normal representation of M on $\mathcal{H}$ and $\xi \in \mathcal{H}$, then $x \to \langle \pi(x)\xi, \xi \rangle$ is a normal linear functional on M. Thus if ϕ is a positive linear functional on M and π_ϕ is normal, then ϕ is normal.

p. 252 III.2.2.15: Some of the assertions need to be clarified and justified. First, the fact that an infinite-dimensional von Neumann algebra M has nonnormal representations is equivalent to the assertion that M has nonnormal states, i.e. that $M_* \neq M^*$, or that M is not reflexive as a Banach space. This follows from the addendum to p. 69.

Next, if M is a W*-algebra and π is a representation (not assumed to be normal) of M on a separable Hilbert space, then it follows from the results of

[Han81] that $\pi(M)$ is an AW*-algebra (it appears to be unknown whether it must be a W*-algebra). The image is not weakly closed in general (see below), but it is, for example, if M is a factor and π is faithful, by III.1.8.6; in this case π is necessarily normal (III.2.2.1). See [**?**] for related results.

We can say more in the factor case. If M is a countably decomposable factor (e.g. a factor with separable predual), then either M is a simple C*-algebra (if M is finite or Type III) or it has exactly one nontrivial norm-closed ideal J (if M is Type I_∞ or Type II_∞) by III.1.7.11. In the latter case, M/J is a simple unital C*-algebra which is not countably decomposable (III.1.7.12 in the addendum to p. 243). Thus M/J has no nonzero representation on a separable Hilbert space. So, in any case, any nonzero representation of M on a separable Hilbert space must be faithful, and hence normal by the previous paragraph.

If M is a factor which is not countably decomposable (or, more generally, a von Neumann algebra with no nonzero countably decomposable central projection), then M contains an uncountable set of mutually orthogonal projections each equivalent to the identity, so no quotient of M is countably decomposable. Thus M can have no nonzero representations (normal or not) on a separable Hilbert space.

Note that a representation of a general W*-algebra on a separable Hilbert space is not necessarily normal. For example, $l^\infty \cong C(\beta\mathbb{N})$ has nonnormal pure states (any pure state coming from a point of $\beta\mathbb{N} \setminus \mathbb{N}$) and hence one-dimensional nonnormal representations. (In fact, almost all pure states of infinite-dimensional W*-algebras are nonnormal.) If π is the sum of the identity representation of l^∞ and such a one-dimensional representation on $\mathcal{H} = l^2 \oplus \mathbb{C}$, then $\pi(l^\infty)$ is not weakly closed: if p_n is the projection in l^∞ consisting of n 1's followed by 0's, then the supremum of $\{\pi(p_n)\}$ in $\mathcal{L}(\mathcal{H})$ is the projection Q from $\mathcal{H}$ onto l^2, so $Q \in \pi(l^\infty)''$, but $Q \notin \pi(l^\infty)$.

Also, the quotient of a W*-algebra by a norm-closed ideal is not necessarily an AW*-algebra: for example, the Calkin algebra is not an AW*-algebra (for one thing, it has nontrivial K_1). Also, l^∞/c_o is not an AW*-algebra since $\beta\mathbb{N} \setminus \mathbb{N}$ is not extremally disconnected [GJ76,6R]. These are examples of corona algebras or SAW*-algebras [**?**].

p. 265 III.2.5.20: See [**?**], [**?**], and [**?**] for more results on groups with simple reduced C*-algebras.

p. 281 III.3.2.2(iii): "norm-continuous" means "jointly continuous when N_* has the norm topology."

p. 283 III.3.2.6 line -5: $t \mapsto u_t \in N \bar{\rtimes}_\alpha G$

p. 285 III.3.2.14 line -5: "Let μ_i be a probability measure on X_i ..."

p. 318 III.5.2.2: A C*-algebra has separable dual if and only if it has a countable composition series in which the successive quotients are separable elementary

C*-algebras; these are precisely the separable scattered C*-algebras (cf. the addendum to p. 69).

p. 327 IV.1.2.7: cf. [Dx69b,4.7.20]. Such a C*-algebra is sometimes called a *dual C*-algebra*, although this terminology is not consistent with III.2.4.2. A dual C*-algebra is scattered (cf. the addenda to p. 69 and p. 318).

p. 331 IV.1.4.3: The proof actually shows more than the stated result. The Proposition should be rephrased:

PROPOSITION. Let A be a C*-algebra [*resp.* unital C*-algebra]. The following are equivalent:

(i) A is subhomogeneous.

(ii) A is isomorphic to a C*-subalgebra of a homogeneous C*-algebra.

(iii) A is isomorphic to a C*-subalgebra [*resp.* unital C*-subalgebra] of a unital homogeneous C*-algebra.

(iv) A is isomorphic to a C*-subalgebra [*resp.* unital C*-subalgebra] of $C(T, \mathbb{M}_k)$ for some k and some compact Hausdorff space T.

The proof in the book shows (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iv), and (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) is trivial.

p. 335 IV.1.4.17: The converse is clear: Fell's condition is equivalent to being of Type I_0 [**?**]. In fact, A is Type I_0 if and only if it is a sum of continuous trace ideals, i.e. if and only if $\hat{A}$ is a union of open subsets corresponding to continuous trace ideals. In particular, if A is Type I_0, then every point of $\hat{A}$ has a Hausdorff neighborhood.

p. 335 IV.1.4.18: in the proof, (i) $\Longleftrightarrow$ (ii) should be (i) $\Longrightarrow$ (ii)

p.335 line -9: "… Ap is an $A - C(T)$-imprimitivity bimodule."

p. 353 IV.2.1.7: see the errata for p. 244 III.1.8.6.

p. 357 IV.2.2.8: The proof that the set of unital completely positive maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$ is compact in the point-σ-weak topology is a standard argument used, for example, in the proof of Alaoglu's theorem (cf. the proof of IV.1.4.3); we outline the argument. The result is more general:

PROPOSITION. Let A be a unital C*-algebra and M a von Neumann algebra, and $CP_1(A, M)$ the set of unital completely positive maps from A to M. Then $CP_1(A, M)$ is compact in the point-σ-weak topology.

PROOF: For each $x \in A$, let $\mathcal{B}_x$ be the closed ball around 0 in M of radius $\|x\|$. Then $\mathcal{B}_x$ is σ-weakly compact by I.3.2.2, and hence $\mathcal{B} = \prod_{x\in A} \mathcal{B}_x$ is compact in the product topology. $\mathcal{B}$ can be identified with the set of functions $\theta : A \to M$

with the property that $\|\theta(x)\| \leq \|x\|$ for all $x \in A$; the topology on functions is the point-σ-weak topology. By II.6.9.4, $CP_1(A, M)$ is naturally a subset of $\mathcal{B}$, and it is completely routine and straightforward (using I.3.1.4 and II.6.9.4) to check that $CP_1(A, M)$ is closed in $\mathcal{B}$.

p. 365 IV.2.5.1: It should be pointed out that a von Neumann algebra which is amenable as a von Neumann algebra is not amenable as a C*-algebra in general. In fact, the only von Neumann algebras which are amenable as C*-algebras are the von Neumann algebras which are Type I C*-algebras (IV.1.1.5). See the addendum to p. 384.

p. 368 IV.3.1.1: See the addendum to p. 182 for the definition of the map from $(A \otimes_{\min} B) \otimes_{\max} C$ to $A \otimes_{\min} (B \otimes_{\max} C)$.

p. 379 IV.3.3.2: The formula for the virtual diagonal d is incorrect, and thus the formula for the x is also incorrect. The correct formulas are:

$$d = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} \otimes e_{ji}$$

$$x = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} \delta(e_{ji}) \ .$$

p. 384: It is a corollary of IV.3.4.3 and the fact that $C^*(\mathbb{F}_2)$ is not exact (II.9.6.6) that a von Neumann algebra which is not a Type I C*-algebra (cf. IV.1.1.5) is not an exact C*-algebra (and in particular not a nuclear C*-algebra). Since $C^*(\mathbb{F}_2)$ is residually finite-dimensional, it embeds in a full direct product of matrix algebras; such a full direct product embeds in any von Neumann algebra which is not Type I of bounded degree.

p. 391 IV.3.5.1: The right-hand vertical map in the diagram should be $\pi \circ \rho$, where ρ is the map from $(A \otimes_{\max} B) \rtimes_\beta G$ to $(A \otimes_{\min} B) \rtimes_\beta G$ induced by the β-equivariant map from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ (cf. II.10.3.21 in the addendum to p. 205).

p. 421 V.2.1.11: Free groups are quite unique in this regard; see [**?**].

p. 445 V.3.1.1: If X is a compact Hausdorff space, the number defined in the statement of the theorem coincides with the covering dimension of X even if X is not metrizable; however, this number may not agree with other types of dimensions for X in the nonmetrizable case.

p. 448 V.3.1.12: T should be $\mathcal{T}$.

p. 452 lines 10-11: calculus

p. 452 V.3.2.2: The formula only follows from V.3.1.1 if X is metrizable. But $rr(C(X))$ coincides with the covering dimension of X for arbitrary compact Hausdorff X (see the addendum to V.3.1.1).

p. 467 V.4.3.6(iii): add (V.4.2.11).

p. 475 V.4.3.37: The condition that the image of ϕ hit the interior of every face in K is too strong and unnecessary. We only require that every vertex of K be in the image, and that for every set of vertices spanning a face of K, there be a point in the image for which all coordinates corresponding to these vertices are positive (i.e. there is no proper subcomplex of K containing the image). Not every partition of unity or complete order embedding of $\mathbb{C}^n$ corresponds to a weak triangulation hitting the interior of every face. The condition of hitting the interior of every face can always be arranged, but the corresponding conditions on the partitions of unity or the complete order embeddings are awkward to state cleanly.

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