Semiprojectivity: a Retrospective

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Absolute Retracts

Absolute Retracts

We will work in the category of *compact metrizable spaces*.

"space" = "compact metrizable space."
"map" = "continuous function."

Definition

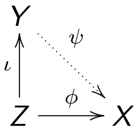
A space X is an *absolute retract (AR)* if, whenever X is a subspace of a space Y, there is a retraction from Y onto X.

Absolute Retracts

Absolute retracts have a stronger property:

Theorem.

A space X is an AR if and only if, whenever Y is a space and Z a subspace of Y, and $\phi : Z \to X$ is a map, there is an extension of ϕ to a map $\psi : Y \to X$. (X has the *extension property*.)



An AR is an injective object in the category of spaces and maps.

Absolute Retracts

Any space with the extension property is an AR.

The Tietze Extension Theorem says that $[0, 1]$ has the extension property (and is thus an AR).

Any product of spaces with the extension property has the extension property. In particular, any cube (e.g. the Hilbert cube) has the extension property.

Any retract of a space with the extension property has the extension property.

Any AR is a retract of the Hilbert cube and thus has the extension property.

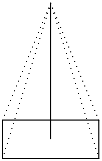
Absolute Retracts

An absolute retract is contractible. More is true:

Proposition

Let X be an AR, and $p \in X$. Then X is contractible to $\{p\}$ relative to $\{p\}$.

Let
$Y = CX = (X \times [0,1])/(X \times \{1\}), Z = (X \times \{0\}) \cup (\{p\} \times [0,1])$.

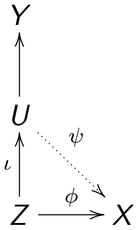


Absolute Neighborhood Retracts

Absolute Neighborhood Retracts

Definition.

A space X is an *Absolute Neighborhood Retract (ANR)* if, whenever Z is a subspace of a space Y and $\phi : Z \to X$ a map, there is an extension of ϕ to a map ψ from some neighborhood U of Z to X.



Absolute Neighborhood Retracts

Absolute Neighborhood Retracts are "locally AR", i.e. spaces with nice local structure (e.g. locally contractible). The converse is also true. So, for example, every polyhedron is an ANR.

A contractible ANR is an AR. So the obstruction to an ANR being an AR is global nontriviality of the topology (e.g. nontrivial homology).

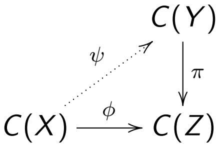
ANR's are important in many aspects of topology, such as shape theory. Every space can be written as an inverse limit of ANR's, "uniquely up to homotopy."

Turning the Arrows Around

In order to extend the notions of AR and ANR to the noncommutative case, we need to rephrase the definitions in terms of the C*-algebra $C(X)$. This is straightforward for an AR:

Proposition

Let X be a space. Then X is an AR if and only if, for any spaces Y and Z and surjective unital *-homomorphism $\pi : C(Y) \to C(Z)$, and any unital *-homomorphism $\phi : C(X) \to C(Z)$, there is a unital *-homomorphism $\psi : C(X) \to C(Y)$ such that $\phi = \pi \circ \psi$ (we say ϕ *lifts* to Y).



Turning the Arrows Around

$C(X)$ is a projective object in the category of unital commutative separable C*-algebras and unital *-homomorphisms.

The ANR case is not so straightforward. The key observation is:

Proposition.

Let X be a space. Then X is an ANR if and only if, whenever Y is a space, (Z_n) a decreasing sequence of subspaces with $Z = \cap_n Z_n$, and $\phi : Z \to X$ a map, then ϕ extends to a map $\psi : Z_n \to X$ for some sufficiently large n.

Projectivity and Semiprojectivity

We are then led to the fundamental definition by turning arrows around:

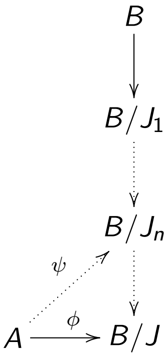
Definition.

Let **C** be a category of C*-algebras. A separable C*-algebra $A \in \mathbf{C}$ is *semiprojective* (in **C**) if, whenever $B \in \mathbf{C}$, (J_n) is an increasing sequence of (closed two-sided) ideals of B with $J = [\cup_n J_n]^-$, and $\phi : A \to B/J$ is a *-homomorphism in **C**, then there is a *partial lift* $\psi : A \to B/J_n$ for some sufficiently large n, with ψ a *-homomorphism in **C** satisfying $\psi = \pi \circ \phi$, where $\pi : B/J_n \to B/J$ is the quotient map.

If ϕ always lifts to $\psi : A \to B$, then A is *projective* in **C**.

The categories we will mainly consider are $\mathbf{C}^*$, the category of all separable C*-algebras and *-homomorphisms, and $\mathbf{C}_1^*$, the category of separable unital C*-algebras and unital *-homomorphisms.

Projectivity and Semiprojectivity



The name "semiprojective" is appropriate, but its use here was a mistake: Effros and Kaminker previously used the term for a different, less restrictive condition.

Projectivity and Semiprojectivity

Note that semiprojectivity is only defined for separable C*-algebras.

Elementary observations:
The definition does not change if we make any or all of the following restrictions:

B is separable. (From now on, most C*-algebras will be separable.)

ϕ is injective.

ϕ is surjective.

The Homotopy Lifting Property

Semiprojective C*-algebras have an important lifting property with respect to inductive limits:

Theorem.

Let $(B_n, \beta_{n,m})$ be an inductive system of C*-algebras with not necessarily injective connecting maps, with inductive limit B and induced *-homomorphisms $\beta_n : B_n \to B$. Let A be a semiprojective C*-algebra and $\phi : A \to B$ a *-homomorphism. Then, for all sufficiently large n, there is a *-homomorphism $\phi_n : A \to B_n$ such that $\beta_n \circ \phi_n$ is homotopic to ϕ as *-homomorphisms from A to B, and $\beta_n \circ \phi_n \to \phi$ in the point-norm topology.

The statement and proof are exactly analogous to those of a corresponding standard result in topology, using a mapping telescope construction.

The Homotopy Lifting Property

The defining property of semiprojectivity is a stronger version of the Homotopy Lifting Property for inductive systems with *surjective* connecting maps.

Semiprojectivity and the Homotopy Lifting Property can be thought of in terms of *stable relations*. If $A = C^*(\mathcal{G}|\mathcal{R})$ is the universal C*-algebra on a set $\mathcal{G}$ of generators and a set $\mathcal{R}$ of relations, and A is semiprojective, then whenever a C*-algebra B contains elements approximately satisfying the relations, the elements can be slightly perturbed to exactly satisfy the relations.

The Homotopy Lifting Property can be phrased:

If $A = C^*(\mathcal{G}|\mathcal{R})$ is the universal C*-algebra on a set $\mathcal{G}$ of generators and a set $\mathcal{R}$ of relations, a set of generators in $B = \varinjlim B_n$ satisfying the relations can be continuously slightly perturbed so as to lie in B_n for large n, with the perturbed generators exactly satisfying the relations at each time.

The Homotopy Lifting Property

This property has a number of other important consequences, for example:

Corollary.

Let A be a semiprojective C*-algebra which can be written as an inductive limit of C*-algebras with finitely generated K-theory. Then $K_*(A)$ is finitely generated.

This corollary applies to semiprojective Kirchberg algebras.

Question.

Can every separable C*-algebra be written as an inductive limit of C*-algebras with finitely generated K-theory?

The Homotopy Lifting Property

The Homotopy Lifting Theorem has a version for generalized inductive limits (B-Kirchberg). Using this, we get:

Corollary.

A semiprojective MF algebra is residually finite-dimensional. In particular, a quasidiagonal semiprojective C*-algebra is residually finite-dimensional.

Here is a related result of relevance to E-theory, which is easily proved directly from the definition of semiprojectivity:

Proposition.

Let A and B be C*-algebras, with A semiprojective. Then the natural map from the set $[A, B]$ of homotopy classes of *-homomorphisms from A to B, to the set $[[A, B]]$ of homotopy classes of asymptotic homomorphisms from A to B, is a bijection.

Elementary Examples of Semiprojective C*-Algebras

Proposition.

$\mathbb{C}$ is semiprojective in $\mathbf{C}^*$.

This is not entirely trivial. It is trivial that $\mathbb{C}$ is projective in $\mathbf{C}_1^*$. But $\mathbb{C}$ is not projective in $\mathbf{C}^*$ since projections do not lift from quotients in general. The proof that $\mathbb{C}$ is semiprojective in $\mathbf{C}^*$ is a functional calculus argument.

Corollary.

A unital C*-algebra A is semiprojective in $\mathbf{C}^*$ if and only if it is semiprojective in $\mathbf{C}_1^*$.

If A is semiprojective in $\mathbf{C}_1^*$ and $B, (J_n), J, \phi$ are a system in $\mathbf{C}$, partially lift the unit of A to a projection in B/J_n and cut down.

Elementary Examples of Semiprojective C*-Algebras

We get two other partial lifting results by similar elementary arguments:

Proposition.

Let $B, (J_n), J$, be as in the definition of semiprojectivity, and let q_1 and q_2 be orthogonal projections in B/J. Suppose there is a projection $p_1 \in B/J_n$ for some n with $\pi(p_1) = q_1$. Then, after increasing n if necessary, there is a projection $p_2 \in B/J_n$ with $p_2 \perp p_1$ and $\pi(p_2) = q_2$.

Proposition.

Let $B, (J_n), J$, be as in the definition of semiprojectivity, and let v be a partial isometry in B/J with $v^*v = q_1$, $vv^* = q_2$. If p_1, p_2 are projection preimages for q_1, q_2 in B/J_n for some n, then, after increasing n if necessary, there is a partial isometry preimage u of v in B/J_n with $u^*u = p_1$, $uu^* = p_2$.

Elementary Examples of Semiprojective C*-Algebras

By repeated mindless application of these propositions, we get the following examples of semiprojective C*-algebras:

Any matrix algebra, and more generally any finite-dimensional C*-algebra

$C(\mathbb{T})$, the universal C*-algebra generated by one unitary

$C^*(\mathbb{F}_n)$, the universal C*-algebra generated by n unitaries (n finite!)

The Toeplitz algebra $\mathcal{T}$

The Cuntz algebras O_n (n finite!)

The Cuntz-Krieger algebras O_A, A a finite matrix

Elementary Examples of Semiprojective C*-Algebras

The applications are not as mindless as they seem, however. Consider the following assertion:

Claim.

Let A be a C*-algebra generated by finitely many partial isometries, with finitely many relations, each of which is an order or orthogonality relation among the source and range projections. Then A is semiprojective.

This claim is much more subtle than it first appears. We will return to this matter later.

Elementary Examples of Semiprojective C*-Algebras

The Propositions also rather mindlessly imply:

Proposition.

Let A_1 and A_2 be unital semiprojective C*-algebras. Then $A_1 \oplus A_2$ is semiprojective.

Proposition.

Let A be a unital semiprojective C*-algebra. Then $M_n(A)$ is semiprojective for all n.

These results are also true in the nonunital case, but are considerably harder (Loring).

Elementary Examples of Semiprojective C*-Algebras

Proposition.

A unital full corner in a semiprojective C*-algebra is semiprojective.

Combining the last two results, we obtain:

Corollary.

If A and B are unital C*-algebras which are Morita equivalent (stably isomorphic), and A is semiprojective, so is B.

This is false in general for nonunital C*-algebras: $\mathbb{K}$ is not semiprojective. (No infinite-dimensional simple AF algebra is semiprojective.)

Elementary Nonexamples of Semiprojective C*-Algebras

The mindless proofs do not work for some obvious infinitely-generated examples, such as $C^*(\mathbb{F}_\infty)$ and O_∞, since the n might need to be increased each time another generator is partially lifted. Are these semiprojective?

$C^*(\mathbb{F}_\infty)$ has infinitely generated K-theory, and it obviously is an inductive limit of C*-algebras with finitely generated K-theory. So $C^*(\mathbb{F}_\infty)$ is not semiprojective.

O_∞ has finitely generated K-theory. It turns out that O_∞ *is* semiprojective.

Elementary Nonexamples of Semiprojective C*-Algebras

There is a quite different class of nonexamples. We have that $C(X)$ is semiprojective in the category of (unital) commutative C*-algebras if and only if X is an ANR. But such a $C(X)$ is not obviously semiprojective in the category of general (noncommutative) C*-algebras. In fact, commutation relations are very hard to partially lift:

Proposition.

$C(\mathbb{T}^2)$, the universal C*-algebra generated by two commuting unitaries, is not semiprojective in $\mathbf{C}$.

The Voiculescu matrices can be used to construct an inductive limit where the homotopy lifting property fails. The argument can be modified to show that $C([0,1]^2)$ is not semiprojective.

Elementary Examples and Nonexamples

Conjecture.

Let X be a compact metrizable space. Then $C(X)$ is semiprojective if and only if X is an ANR and $dim(X) \leq 1$.

It is known that if $C(X)$ is semiprojective, then $dim(X) \leq 1$ (and X is an ANR). If X is an AR and $dim(X) \leq 1$, then $C(X)$ is projective in $\mathbf{C}_1^*$, hence semiprojective. If X is a finite graph, then $C(X)$ is semiprojective.

Relative and Sequential Semiprojectivity

Let us regularize the technique used in the mindless arguments.

Definition.

Let A be a separable C*-algebra and D a C*-subalgebra of A. A is *conditionally semiprojective* with respect to D if, for any C*-algebra B, increasing sequence $\langle J_n \rangle$ of (closed two-sided) ideals of B, with $J = [\cup J_n]^-$, and *-homomorphism $\phi : A \to B/J$, and *-homomorphism $\theta : D \to B/J_m$ for some m with $\pi \circ \theta = \phi|_D$, there is an $n \geq m$ and a *-homomorphism $\psi : A \to B/J_n$ such that $\phi = \pi \circ \psi$ and $\psi|_D = \pi_n \circ \theta$.

In other words, a partial lift of $\phi|_D$ can be extended to a partial lift of all of ϕ, at the price of possibly increasing the n. A C*-algebra A is semiprojective if and only if it is conditionally semiprojective with respect to the 0 C*-subalgebra.

Relative and Sequential Semiprojectivity

Definition.

Let A be a C*-algebra with a specified ordered finite set $\mathcal{G} = \{a_1, \ldots, a_r\}$ of generators. Set $A_0 = \{0\}$, and for $1 \leq k \leq r$ let A_k be the C*-subalgebra of A generated by $\{a_1, \ldots, a_k\}$. Then A is *sequentially semiprojective* with respect to $\mathcal{G}$ if A_k is conditionally semiprojective with respect to A_{k-1} for all k, $1 \leq k \leq r$.

Note that the sequential semiprojectivity of a C*-algebra with respect to a set of generators potentially depends on the ordering specified; there are examples where this dependence is more than potential.

Relative and Sequential Semiprojectivity

There is one technicality which is usually needed in the unital case. If A is a unital C*-algebra, and B, J_n, J as in the definition, and $\phi : A \to B/J$ is a *-homomorphism, set $q = \phi(1)$ and partially lift q to a projection $p \in B/J_m$ for some m. Then replace B/J_n for $n \geq m$ by $\pi_n(p)(B/J_n)\pi_n(p)$ and B/J by $q(B/J)q$ to reduce to a unital lifting problem. We will call this step the *preliminary step* and use it without comment in the unital case (it amounts to adding 1 to the beginning of the ordered list of generators). The preliminary step is not always necessary, but it is natural in situations where 1 appears in the relations satisfied by the generators. The preliminary step can be formalized by:

Proposition.

Let A be a unital C*-algebra. Then A is semiprojective if and only if A is conditionally semiprojective with respect to $\mathbb{C}1$.

Relative and Sequential Semiprojectivity

Examples

Let $A = \mathbb{M}_n$. Take as the generating set $\{e_{11}, \ldots, e_{1n}\}$, where $\{e_{ij}\}$ is a set of matrix units in A. Successively partially lift each e_{1k} by first partially lifting its range projection and then partially lifting e_{1k} to one with the specified source and range projection. Thus A is sequentially semiprojective with respect to these generators. (Actually, any finite-dimensional C*-algebra is sequentially semiprojective with respect to any finite set of generators in any order.)

$C^*(\mathbb{F}_n)$ is sequentially semiprojective with respect to the usual sequence of generators (the preliminary step must be done first).

Relative and Sequential Semiprojectivity

Similarly, if $\{s_1, \ldots, s_n\}$ are the standard generators of O_n, after the preliminary step successively partially lift each s_k by first partially lifting the range projection $s_k s_k^*$ to be orthogonal to the sum of the previous range projections (at the last step take the partial lift to be the complement of the sum of the other range projections). Thus O_n is sequentially semiprojective with respect to the generators $\{1, s_1, \ldots, s_n\}$. One could also partially lift all the range projections first, and then the isometries.

Slightly more care must be exercised for the Cuntz-Krieger algebras. If $\{s_1, \ldots, s_n\}$ are the generating partial isometries for O_A, to partially lift s_k properly it might be necessary to first partially lift the range projections of some of the succeeding s_j in order to have a partial lift of the source projection of s_k. But this can be done, so O_A is sequentially semiprojective with respect to the ordered set of generators $\{1, s_1, \ldots, s_n\}$.

The Claim Revisited

The order of the generators does matter:

Let A be the universal C*-algebra generated by three projections $\{p, q, r\}$, where the only relations are that $r \perp p$ and $r \perp q$. It is easily seen that A is semiprojective; in fact, A is sequentially semiprojective with respect to the generators $\{r, p, q\}$ (in this order).

However, A is not sequentially semiprojective with respect to $\{p, q, r\}$ (in this order); specifically, A is not conditionally semiprojective with respect to the C*-subalgebra D generated by p and q.

Note first that D is isomorphic to the universal C*-algebra C generated by two projections p_1, p_2 (there is a left inverse γ for the natural universal map from this universal C*-algebra to D sending p_1 and p_2 to p and q respectively, defined by $\gamma(p) = p_1$, $\gamma(q) = p_2$, $\gamma(r) = 0$).

The Claim Revisited

An example of a partial lift of D which cannot be "extended" to a partial lift of A can be given as follows. Let B be the C*-algebra of convergent sequences of 2×2 matrices, J_n the sequences which are 0 after the n'th term, and J the sequences converging to 0. Then $B/J \cong \mathbb{M}_2$. Let $\phi : A \to B/J$ be defined by $\phi(p) = \phi(q) = diag(1,0)$, $\phi(r) = diag(0,1)$. Lift $\phi(p)$ to the constant sequence $diag(1,0)$ in B, and lift $\phi(q)$ to the sequence whose n'th term is

$$\begin{bmatrix} 1 - \frac{1}{n} & \sqrt{\frac{1}{n} - \frac{1}{n^2}} \\ \sqrt{\frac{1}{n} - \frac{1}{n^2}} & \frac{1}{n} \end{bmatrix} .$$

Then there is no partial lifting of $\phi(r)$ which is orthogonal to the liftings of both $\phi(p)$ and $\phi(q)$, for any n.

The Claim Revisited

The second example, while quite similar, is more dramatic.

Definition.

Let Q_{mn} be the universal C*-algebra generated by an $m \times n$ array of projections $\{q_{ij}\}$ subject to the *array condition*: for fixed i, or for fixed j, the projections q_{ij} are orthogonal.

Thus Q_{22} is the universal C*-algebra generated by four projections $\{q_{ij} : i,j = 1,2\}$ such that $q_{11} \perp q_{12}$, $q_{11} \perp q_{21}$, $q_{22} \perp q_{12}$, $q_{22} \perp q_{21}$, but no relation is assumed between q_{11} and q_{22}, or between q_{12} and q_{21}.

We will show that Q_{22} is not conditionally semiprojective with respect to the C*-subalgebra D generated by $\{q_{11}, q_{12}, q_{21}\}$. Since all four projections are conjugate under automorphisms of Q_{22}, this will show that Q_{22} is not sequentially semiprojective with respect to these four generating projections in any order.

The Claim Revisited

First note that the C*-subalgebra generated by q_{12} and q_{21} is isomorphic to the universal C*-algebra generated by two projections (as is the C*-subalgebra generated by q_{11} and q_{22}), and in fact the C*-subalgebra D is isomorphic to the C*-algebra A of the last example (there is a homomorphism from Q_{22} onto A sending q_{12} to p, q_{21} to q, q_{11} to r, and q_{22} to 0, which is a left inverse for the natural map from A onto D).

One partial lifting problem which cannot be solved is almost identical to the previous one. Let B, J_n, J be as in the definition. Define $\phi : Q_{22} \to B/J \cong \mathbb{M}_2$ by $\phi(q_{11}) = 0$, $\phi(q_{12}) = \phi(q_{21}) = diag(1, 0)$, and $\phi(q_{22}) = diag(0, 1)$. Lift $\phi(q_{11})$ to 0 and $\phi(q_{12})$ and $\phi(q_{21})$ to the previous liftings of p and q. Then there is no partial lifting of $\phi(q_{22})$ which is orthogonal to the liftings of both $\phi(q_{12})$ and $\phi(q_{21})$, for any n.

The Claim Revisited

Note that the C*-algebra D is also not conditionally semiprojective with respect to the C*-subalgebra C generated by q_{12} and q_{21}; this is the previous example. Similarly, Q_{22} is not conditionally semiprojective with respect to the C*-subalgebra C: a lifting problem which cannot be solved is essentially the same as the one above. By symmetry, Q_{22} is also not conditionally semiprojective with respect to the C*-subalgebra generated by q_{11} and q_{22}.

Nonetheless, we have:

Proposition.

Q_{22} is semiprojective.

The Claim Revisited

The proof is quite simple, but nonobvious. Set $c = q_{11} + q_{22}$ and $d = q_{12} + q_{21}$. Then c and d are positive elements of Q_{22} (but not projections). Since $q_{12} \perp q_{11}$ and $q_{12} \perp q_{22}$, we have $q_{12} \perp c$, and similarly $q_{21} \perp c$; hence $d \perp c$.

Suppose B, J_n, J, and ϕ are as in the definition. First lift $\phi(c)$ and $\phi(d)$ to orthogonal positive elements a and b in B: set $y = \phi(c) - \phi(d)$. Then $y = y^* \in B/J$, and $\phi(c)$ and $\phi(d)$ are the positive and negative parts y_+ and y_- of y respectively. Let $x = x^* \in B$ with $\pi(x) = y$, and set $a = x_+$, $b = x_-$. We have $a \perp b$, and $\pi(a) = \pi(x_+) = y_+ = \phi(c)$ and $\pi(b) = \phi(d)$ since x_+, x_- are formed from x by functional calculus.

The Claim Revisited

Now replace B by the hereditary C*-subalgebra generated by a, and similarly for B/J_n, B/J. (The π_n are surjective.) We can thus partially lift $\phi(q_{11})$ and $\phi(q_{22})$ to projections in the hereditary C*-subalgebra of B/J_n generated by $\pi_n(a)$, for some n. Similarly, increasing n if necessary, we can partially lift $\phi(q_{12})$ and $\phi(q_{21})$ to projections in the hereditary C*-subalgebra of B/J_n generated by $\pi_n(b)$. Since these hereditary C*-subalgebras are orthogonal, we get the desired relations among the lifted projections.

Note that Q_{22} is actually the direct sum of the hereditary C*-subalgebras generated by c and d, each of which is isomorphic to the universal C*-algebra generated by two projections. Thus semiprojectivity of Q_{22} also follows from the results of Loring.

Question.

What about other Q_{mn}?

The Claim Revisited

We now discuss a considerably more complex example. Let Q_{mn}^r be the universal C*-algebra generated by an $m \times n$ array of projections satisfying the array condition and for which the rows sum to the identity. Then $m \leq n$, and if $m = n$ the columns also sum to the identity. Write $A_s(n)$ for Q_{nn}^r; these are an interesting class of quantum groups (Wang, Banica, ...)

Proposition.

If Q_{mn} is semiprojective, so is Q_{mn}^r.

So $A_s(4)$ is the universal unital C*-algebra generated by a 4×4 array of projections $\{p_{ij}\}$ such that the projections in each row and in each column add to the identity.

It is easy to show that $A_s(4)$ is not sequentially semiprojective with respect to the generators $\{p_{ij}\}$ in any order.

The Claim Refuted(?)

Conjecture.

$A_s(4)$ is not semiprojective.

$A_s(4)$ is 4-subhomogeneous: it is isomorphic to a unital C*-subalgebra of $C(S^3, \mathbb{M}_4)$ (Banica-Collins). In particular, there is a homomorphism from $A_s(4)$ onto $C([0,1]^3, \mathbb{M}_4)$. It appears that $C(X)$ is a recursive subhomogeneous C*-algebra.

A standard example of an inductive system not satisfying the Homotopy Lifting Property for $C([0,1]^3, \mathbb{M}_4)$ should also not satisfy the property for $A_s(4)$. (More generally, a subhomogeneous C*-algebra with spectrum more than one-dimensional should not be semiprojective.)

Thus Q_{44} is should also not be semiprojective. It is then unlikely that any Q_{mn} is semiprojective unless $m = n = 2$.

Semiprojective Simple C*-Algebras

It is hard for a simple C*-algebra to be semiprojective. The only obvious examples are:

Finite-dimensional matrix algebras

The Cuntz algebras O_n (n finite) and the simple Cuntz-Krieger algebras O_A.

There is one other familiar semiprojective simple C*-algebra:

Theorem.

O_∞ is semiprojective.

Semiprojective Simple C*-Algebras

The proof is a modification of a sequential argument. Once two generators have been partially lifted to B/J_n, the rest can be partially lifted without further increasing n. The lifting is not truly sequential since at each stage, in order to lift the next generator, the lift of the previous generator must be modified.

The fact that makes the argument work is that a unitary in the connected component of the identity in a quotient can always be lifted.

The technique can be pushed a little farther, but there are much more extensive results of Szymanski and Spielberg. The best known result is:

Definition.

A *Kirchberg algebra* is a separable simple nuclear purely infinite C*-algebra in the UCT class.

Theorem.

Let A be a Kirchberg algebra with finitely generated K-theory, with $K_1(A)$ torsion-free. Then A is semiprojective.

The results so far suggest a conjecture:

Conjecture.

Every Kirchberg algebra with finitely generated K-theory is semiprojective.

Note that the finite generation hypothesis for K-theory is necessary.

Semiprojective Simple C*-Algebras

The most important test case for the conjecture is $A = O_n \otimes O_n$, the Kirchberg algebra with $K_0(A) = K_1(A) = \mathbb{Z}_{n-1}$.

It could be argued that the natural set of generators and relations include commutation relations, which are very hard to partially lift, so the tensor product is unlikely to be semiprojective.

However, the same argument could be made for $O_n \otimes O_m$ for any n and m. But it is a (deep) fact that if $n-1$ and $m-1$ are relatively prime, then $O_n \otimes O_m \cong O_2$ and is therefore semiprojective.

My guess is that $O_n \otimes O_n$ is semiprojective, but any proof will have to use deep structure properties.

Semiprojective Simple C*-Algebras

There is reason to believe the following conjecture:

Conjecture.

The only nuclear semiprojective simple unital C*-algebras are the finite-dimensional matrix algebras and the Kirchberg algebras with finitely generated K-theory.

An infinite-dimensional semiprojective simple nuclear C*-algebra cannot be quasidiagonal. Hence infinite-dimensional finite simple nuclear C*-algebras are unlikely to be semiprojective.

It is at least very likely that any infinite-dimensional semiprojective simple C*-algebra is either purely infinite or else projectionless (projectionless corner).

Semiprojective Simple C*-Algebras

There is evidence that every semiprojective simple C*-algebra is at least exact, if not nuclear. The following conjecture of Kirchberg (a variation of the Connes embedding problem) seems reasonable:

Conjecture.

Let B be the C*-algebra of bounded sequences of elements of O_2, and J the ideal of sequences converging to 0. Then every separable C*-algebra embeds into B/J.

If this conjecture is true, it would follow that every semiprojective simple C*-algebra embeds in O_2 and hence is exact.

The stable algebra of a semiprojective C*-algebra is rarely semiprojective. But we have:

Theorem.

If A is a semiprojective simple purely infinite C*-algebra, then $A \otimes \mathbb{K}$ is also semiprojective.

The idea of the proof is that A contains a nicely embedded copy of $A \otimes \mathbb{K}$, which only needs to be "moved out." This can be done by lifting unitaries in the connected component of the identity without increasing the n.

Semiprojective Simple C*-Algebras

One final observation. Since a matrix algebra over a semiprojective C*-algebra is semiprojective, what about a crossed product by a finite cyclic group, which is the "square root" of tensoring with a matrix algebra?

Theorem.

There is a $\mathbb{Z}_2$ action σ on O_2 such that $A = O_2 \rtimes_\sigma \mathbb{Z}_2$ is a Kirchberg algebra whose K-theory is not finitely generated.

In fact, the K-groups of A can independently be any countable abelian torsion groups in which every element has odd order.

Corollary.

A crossed product of a semiprojective C*-algebra by $\mathbb{Z}_2$ is not semiprojective in general.