Recent Results on Semiprojectivity

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Happy 60th, David!

Absolute Retracts

Absolute Retracts

We will work in the category of *compact metrizable spaces*.

"space" = "compact metrizable space."
"map" = "continuous function."

Definition

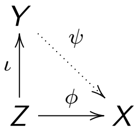
A space X is an *absolute retract (AR)* if, whenever X is a subspace of a space Y, there is a retraction from Y onto X.

Absolute Retracts

Absolute retracts have a stronger property:

Theorem.

A space X is an AR if and only if, whenever Y is a space and Z a subspace of Y, and $\phi : Z \to X$ is a map, there is an extension of ϕ to a map $\psi : Y \to X$. (X has the *(Tietze) extension property*.)



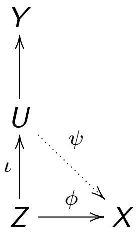
An AR is an injective object in the category of spaces and maps.

Absolute Neighborhood Retracts

Absolute Neighborhood Retracts

Definition.

A space X is an *Absolute Neighborhood Retract (ANR)* if, whenever Z is a subspace of a space Y and $\phi : Z \to X$ a map, there is an extension of ϕ to a map ψ from some neighborhood U of Z to X.



Absolute Neighborhood Retracts

Absolute Neighborhood Retracts are "locally AR", i.e. spaces with nice local structure (e.g. locally contractible). The converse is also true. So, for example, every polyhedron is an ANR.

A contractible ANR is an AR. So the obstruction to an ANR being an AR is global nontriviality of the topology (e.g. nontrivial homology).

ANR's are important in many aspects of topology, such as shape theory. Every space can be written as an inverse limit of ANR's, "uniquely up to homotopy."

Turning the Arrows Around

In order to extend the notions of AR and ANR to the noncommutative case, we need to rephrase the definitions in terms of the C*-algebra $C(X)$. This is straightforward for an AR:

Proposition

Let X be a space. Then X is an AR if and only if, for any spaces Y and Z and surjective unital *-homomorphism $\pi : C(Y) \to C(Z)$, and any unital *-homomorphism $\phi : C(X) \to C(Z)$, there is a unital *-homomorphism $\psi : C(X) \to C(Y)$ such that $\phi = \pi \circ \psi$ (we say ϕ *lifts* to Y).

$$\begin{array}{ccc} & & C(Y) \\ & \overset{\psi}{\nearrow} & \downarrow \pi \\ C(X) & \xrightarrow{\phi} & C(Z) \end{array}$$

Turning the Arrows Around

$C(X)$ is a projective object in the category of unital commutative separable C*-algebras and unital *-homomorphisms.

The ANR case is not so straightforward. The key observation is:

Proposition.

Let X be a space. Then X is an ANR if and only if, whenever Y is a space, (Z_n) a decreasing sequence of subspaces with $Z = \cap_n Z_n$, and $\phi : Z \to X$ a map, then ϕ extends to a map $\psi : Z_n \to X$ for some sufficiently large n.

Projectivity and Semiprojectivity

We are then led to the fundamental definition by turning arrows around:

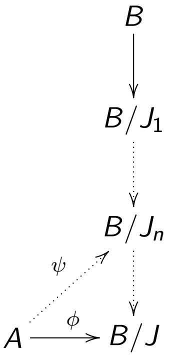
Definition.

Let **C** be a category of C*-algebras. A separable C*-algebra $A \in \mathbf{C}$ is *semiprojective* (in **C**) if, whenever $B \in \mathbf{C}$, (J_n) is an increasing sequence of (closed two-sided) ideals of B with $J = [\cup_n J_n]^-$, and $\phi : A \to B/J$ is a *-homomorphism in **C**, then there is a *partial lift* $\psi : A \to B/J_n$ for some sufficiently large n, with ψ a *-homomorphism in **C** satisfying $\psi = \pi \circ \phi$, where $\pi : B/J_n \to B/J$ is the quotient map.

If ϕ always lifts to $\psi : A \to B$, then A is *projective* in **C**.

The categories we will mainly consider are $\mathbf{C}^*$, the category of all separable C*-algebras and *-homomorphisms, and $\mathbf{C}^*_1$, the category of separable unital C*-algebras and unital *-homomorphisms.

Projectivity and Semiprojectivity



The name "semiprojective" is appropriate, but its use here was a mistake: Effros and Kaminker previously used the term for a different, less restrictive condition.

Projectivity and Semiprojectivity

Note that semiprojectivity is only defined for separable C*-algebras.

Elementary observations:
The definition does not change if we make any or all of the following restrictions:

B is separable. (From now on, most C*-algebras will be separable.)

ϕ is injective.

ϕ is surjective.

Elementary Examples of Semiprojective C*-Algebras

Proposition.

$\mathbb{C}$ is semiprojective in $\mathbf{C}^*$.

This is not entirely trivial. It is trivial that $\mathbb{C}$ is projective in $\mathbf{C}^*_1$. But $\mathbb{C}$ is not projective in $\mathbf{C}^*$ since projections do not lift from quotients in general. The proof that $\mathbb{C}$ is semiprojective in $\mathbf{C}^*$ is a functional calculus argument.

Corollary.

A unital C*-algebra A is semiprojective in $\mathbf{C}^*$ if and only if it is semiprojective in $\mathbf{C}^*_1$.

If A is semiprojective in $\mathbf{C}^*_1$ and $B, (J_n), J, \phi$ are a system in $\mathbf{C}$, partially lift the unit of A to a projection in B/J_n and cut down.

Elementary Examples of Semiprojective C*-Algebras

We get two other partial lifting results by similar elementary arguments:

Proposition.

Let $B, (J_n), J$, be as in the definition of semiprojectivity, and let q_1 and q_2 be orthogonal projections in B/J. Suppose there is a projection $p_1 \in B/J_n$ for some n with $\pi(p_1) = q_1$. Then, after increasing n if necessary, there is a projection $p_2 \in B/J_n$ with $p_2 \perp p_1$ and $\pi(p_2) = q_2$.

Proposition.

Let $B, (J_n), J$, be as in the definition of semiprojectivity, and let v be a partial isometry in B/J with $v^*v = q_1$, $vv^* = q_2$. If p_1, p_2 are projection preimages for q_1, q_2 in B/J_n for some n, then, after increasing n if necessary, there is a partial isometry preimage u of v in B/J_n with $u^*u = p_1$, $uu^* = p_2$.

Elementary Examples of Semiprojective C*-Algebras

By repeated mindless application of these propositions, we get the following examples of semiprojective C*-algebras:

Any matrix algebra, and more generally any finite-dimensional C*-algebra

$C(\mathbb{T})$, the universal C*-algebra generated by one unitary

$C^*(\mathbb{F}_n)$, the universal C*-algebra generated by n unitaries (n finite!)

The Toeplitz algebra $\mathcal{T}$

The Cuntz algebras O_n (n finite!)

The Cuntz-Krieger algebras O_A, A a finite matrix

Elementary Examples of Semiprojective C*-Algebras

The Propositions also rather mindlessly imply:

Proposition.

Let A_1 and A_2 be unital semiprojective C*-algebras. Then $A_1 \oplus A_2$ is semiprojective.

Proposition.

Let A be a unital semiprojective C*-algebra. Then $M_n(A)$ is semiprojective for all n.

These results are also true in the nonunital case, but are considerably harder (Loring).

Elementary Examples of Semiprojective C*-Algebras

Proposition.

A unital full corner in a semiprojective C*-algebra is semiprojective.

Combining the last two results, we obtain:

Corollary.

If A and B are unital C*-algebras which are Morita equivalent (stably isomorphic), and A is semiprojective, so is B.

This is false in general for nonunital C*-algebras: $\mathbb{K}$ is not semiprojective. (No infinite-dimensional AF algebra is semiprojective.)

The Inductive Limit Lifting Property

Semiprojective C*-algebras have an important lifting property with respect to inductive limits:

Theorem.

Let $(B_n, \beta_{n,m})$ be an inductive system of C*-algebras with not necessarily injective connecting maps, with inductive limit B and induced *-homomorphisms $\beta_n : B_n \to B$. Let A be a semiprojective C*-algebra and $\phi : A \to B$ a *-homomorphism. Then, for all sufficiently large n, there is a *-homomorphism $\phi_n : A \to B_n$ such that $\beta_n \circ \phi_n$ is homotopic to ϕ as *-homomorphisms from A to B, and $\beta_n \circ \phi_n \to \phi$ in the point-norm topology.

The statement and proof are exactly analogous to those of a corresponding standard result in topology, using a mapping telescope construction.

The Inductive Limit Lifting Property

The defining property of semiprojectivity is a stronger version of the Inductive Limit Lifting Property for inductive systems with *surjective* connecting maps.

Semiprojectivity and the Inductive Limit Lifting Property can be thought of in terms of *stable relations*. If $A = C^*(\mathcal{G}|\mathcal{R})$ is the universal C*-algebra on a set $\mathcal{G}$ of generators and a set $\mathcal{R}$ of relations, and A is semiprojective, then whenever a C*-algebra B contains elements approximately satisfying the relations, the elements can be slightly perturbed to exactly satisfy the relations.

The Inductive Limit Lifting Property can be phrased:
If $A = C^*(\mathcal{G}|\mathcal{R})$ is the universal C*-algebra on a set $\mathcal{G}$ of generators and a set $\mathcal{R}$ of relations, a set of generators in $B = \varinjlim B_n$ satisfying the relations can be continuously slightly perturbed so as to lie in B_n for large n, with the perturbed generators exactly satisfying the relations at each time.

The Inductive Limit Lifting Property

This property has a number of other important consequences, for example:

Corollary.

Let A be a semiprojective C*-algebra which can be written as an inductive limit of C*-algebras with finitely generated K-theory. Then $K_*(A)$ is finitely generated.

This corollary applies to semiprojective Kirchberg algebras.

Question.

Can every separable C*-algebra be written as an inductive limit of C*-algebras with finitely generated K-theory?

The Inductive Limit Lifting Property

The Inductive Limit Lifting Property has a version for generalized inductive limits (B-Kirchberg). Using this, we get:

Corollary.

A semiprojective MF algebra is residually finite-dimensional. In particular, a quasidiagonal semiprojective C*-algebra is residually finite-dimensional.

Here is a related result of relevance to E-theory, which is easily proved directly from the definition of semiprojectivity:

Proposition.

Let A and B be C*-algebras, with A semiprojective. Then the natural map from the set $[A, B]$ of homotopy classes of *-homomorphisms from A to B, to the set $[[A, B]]$ of homotopy classes of asymptotic homomorphisms from A to B, is a bijection.

The Homotopy Lifting Question

Sometimes "turning the arrows around" does not work. Here is a theorem from the commutative case:

Theorem.

(Borsuk) Let X be an ANR, Y a space, and Z a closed subspace of Y. Let (ϕ_t) $(0 \leq t \leq 1)$ be a path of maps from Z to X. If ϕ_0 extends to a map ψ_0 from Y to X, then the ϕ_t all extend to a path of maps from Y to X.

The statement has a meaningful version for C*-algebras, but the proof does not work; the essential difficulty is that the primitive ideal space of a noncommutative C*-algebra need not be Hausdorff. (The fact that makes the proof work in the commutative case is that every closed set in a compact metrizable space is a G_δ.)

The Homotopy Lifting Question

Question.

If A and B are (separable) C*-algebras with A semiprojective, J is an ideal of B, and (ϕ_t) $(0 \leq t \leq 1)$ is a path of *-homomorphisms from A to B/J, and ϕ_0 lifts to a *-homomorphism ψ_0 from A to B, does the whole path lift to a path (ψ_t) of *-homomorphisms from A to B?

At least some of the ϕ_t lift: there is an $\epsilon > 0$ such that (ϕ_t) $(0 \leq t \leq \epsilon)$ lifts to a path of homomorphisms from A to B. And for special A, e.g. $A = \mathbb{C}$ or $A = C(T)$, the question has a positive answer.

Elementary Nonexamples of Semiprojective C*-Algebras

The mindless proofs do not work for some obvious infinitely-generated examples, such as $C^*(\mathbb{F}_\infty)$ and O_∞, since the n might need to be increased each time another generator is partially lifted. Are these semiprojective?

$C^*(\mathbb{F}_\infty)$ has infinitely generated K-theory, and it obviously is an inductive limit of C*-algebras with finitely generated K-theory. So $C^*(\mathbb{F}_\infty)$ is not semiprojective.

O_∞ has finitely generated K-theory. It turns out that O_∞ *is* semiprojective.

Elementary Nonexamples of Semiprojective C*-Algebras

There is a quite different class of nonexamples. We have that $C(X)$ is semiprojective in the category of (unital) commutative C*-algebras if and only if X is an ANR. But such a $C(X)$ is not obviously semiprojective in the category of general (noncommutative) C*-algebras. In fact, commutation relations are very hard to partially lift:

Proposition.

$C(\mathbb{T}^2)$, the universal C*-algebra generated by two commuting unitaries, is not semiprojective in **C**.

The Voiculescu matrices can be used to construct an unsolvable lifting problem. The argument can be modified to show that $C([0,1]^2)$ is not semiprojective.

Elementary Examples and Nonexamples

A. Sørensen and H. Thiel have recently proved:

Theorem.

Let X be a compact metrizable space. Then $C(X)$ is semiprojective (in the category of general C*-algebras) if and only if X is an ANR and $dim(X) \leq 1$.

This result is not surprising, and special cases were known. But the proof is a substantial argument.

Semiprojective Simple C*-Algebras

Semiprojective Simple C*-Algebras

It is hard for a simple C*-algebra to be semiprojective. The only obvious examples are:

Finite-dimensional matrix algebras

The Cuntz algebras O_n (n finite) and the simple Cuntz-Krieger algebras O_A.

There is one other familiar semiprojective simple C*-algebra:

Theorem.

O_∞ is semiprojective.

Semiprojective Simple C*-Algebras

The proof is a modification of a sequential argument. Once two generators have been partially lifted to B/J_n, the rest can be partially lifted without further increasing n. The lifting is not truly sequential since at each stage, in order to lift the next generator, the lift of the previous generator must be modified.

The fact that makes the argument work is that a unitary in the connected component of the identity in a quotient can always be lifted.

The technique can be pushed a little farther, but there are much more extensive results of Szymanski and Spielberg. The best known result is:

Definition.

A *Kirchberg algebra* is a separable simple nuclear purely infinite C*-algebra in the UCT class.

Theorem.

Let A be a Kirchberg algebra with finitely generated K-theory, with $K_1(A)$ torsion-free. Then A is semiprojective.

The results so far suggest a conjecture:

Conjecture.

Every Kirchberg algebra with finitely generated K-theory is semiprojective.

Note that the finite generation hypothesis for K-theory is necessary.

Semiprojective Simple C*-Algebras

The most important test case for the conjecture is $A = O_n \otimes O_n$, the Kirchberg algebra with $K_0(A) = K_1(A) = \mathbb{Z}_{n-1}$.

It could be argued that the natural set of generators and relations include commutation relations, which are very hard to partially lift, so the tensor product is unlikely to be semiprojective.

However, the same argument could be made for $O_n \otimes O_m$ for any n and m. But it is a (deep) fact that if $n-1$ and $m-1$ are relatively prime, then $O_n \otimes O_m \cong O_2$ and is therefore semiprojective.

My guess is that $O_n \otimes O_n$ is semiprojective, but any proof will have to use deep structure properties.

Semiprojective Simple C*-Algebras

There is reason to believe the following conjecture:

Conjecture.

The only nuclear semiprojective simple unital C*-algebras are the finite-dimensional matrix algebras and the Kirchberg algebras with finitely generated K-theory.

An infinite-dimensional semiprojective simple nuclear C*-algebra cannot be quasidiagonal. Hence infinite-dimensional finite simple nuclear C*-algebras are unlikely to be semiprojective.

It is at least very likely that any infinite-dimensional semiprojective simple C*-algebra is either purely infinite or else projectionless (projectionless corner).

Semiprojective Simple C*-Algebras

There is evidence that every semiprojective simple C*-algebra is at least exact, if not nuclear. The following conjecture of Kirchberg (a variation of the Connes embedding problem) seems reasonable:

Conjecture.

Let B be the C*-algebra of bounded sequences of elements of O_2, and J the ideal of sequences converging to 0. Then every separable C*-algebra embeds into B/J.

This conjecture can be rephrased in an equivalent way:

Conjecture.

Every separable C*-algebra has a quasidiagonal extension by $O_2 \otimes \mathbb{K}$.

Work of Elliott, Kucerovsky, and Ng is relevant for this conjecture.

Suppose the conjecture is true. Let J_n be the set of sequences which are 0 after the n'th term. If A is semiprojective, an embedding of A into B/J lifts to an embedding of A into B/J_n for some n. But $B/J_n \cong B$. If A is simple, combining this embedding with a coordinate map gives an embedding of A into O_2.

Thus, if the conjecture is true, it would follow that every semiprojective simple C*-algebra embeds in O_2 and hence is exact.

Semiprojective Simple C*-Algebras

Since a matrix algebra over a semiprojective C*-algebra is semiprojective, what about a crossed product by a finite cyclic group, which is the "square root" of tensoring with a matrix algebra?

Theorem.

There is a $\mathbb{Z}_2$ action σ on O_2 such that $A = O_2 \rtimes_\sigma \mathbb{Z}_2$ is a Kirchberg algebra whose K-theory is not finitely generated.

In fact, the K-groups of A can independently be any countable abelian torsion groups in which every element has odd order.

Corollary.

A crossed product of a semiprojective C*-algebra by $\mathbb{Z}_2$ is not semiprojective in general.

The stable algebra of a semiprojective C*-algebra is rarely semiprojective. But we have:

Theorem.

If A is a semiprojective simple purely infinite C*-algebra, then $A \otimes \mathbb{K}$ is also semiprojective.

The idea of the proof is that A contains a nicely embedded copy of $A \otimes \mathbb{K}$, which only needs to be "moved out." This can be done by lifting unitaries in the connected component of the identity without increasing the n.

This result holds more generally for A properly infinite and semiprojective.

Extensions of Semiprojective C*-Algebras

Suppose

$$0 \to J \to A \to A/J \to 0$$

is an extension of C*-algebras. If J and A/J are semiprojective, is A semiprojective? What if the extension splits?

It is not easy to give a counterexample. But the topologist's sine curve X gives a commutative counterexample, an extension

$$0 \to C_0((0,1]) \to C(X) \to C([-1,1]) \to 0$$

which even splits, but X is not an ANR.

Extensions of Semiprojective C*-Algebras

What if the quotient is just $\mathbb{C}$? Years ago, I conjectured:

Conjecture.

If J is a semiprojective C*-algebra, and

$$0 \to J \to A \to \mathbb{C} \to 0$$

is a split exact sequence, then A is semiprojective.

In other words, A is obtained by adding to J a single projection in its multiplier algebra.

Special cases of this conjecture have been verified over the years.

Extensions of Semiprojective C*-Algebras

Recently, S. Eilers and T. Katsura disproved the conjecture.

In their example, J is the stable algebra of the universal C*-algebra $\mathcal{T}_2$ generated by two isometries with mutually orthogonal range projections. Since $\mathcal{T}_2$ is semiprojective by mindless arguments, and properly infinite, its stable algebra is also semiprojective. Its multiplier algebra contains a sufficiently bad projection.

This example also shows that a nonunital full corner in a (nonunital) semiprojective C*-algebra need not be semiprojective.

Equivariant Semiprojectivity

Let (A, G, α) be a separable C*-dynamical system, i.e. A is a separable C*-algebra, G a second countable locally compact group, and α a continuous homomorphism from G to $\mathrm{Aut}(A)$.

Definition.

(N. C. Phillips) The system (A, G, α) is *equivariantly semiprojective* (or *G-equivariantly semiprojective* for emphasis) if, whenever (B, G, β) is a C*-dynamical system, (J_n) an increasing sequence of closed G-invariant ideals of B with $J = [\cup J_n]^-$, and ϕ is a G-equivariant homomorphism from A to B/J (with the G-action induced by β), there is a G-equivariant homomorphism ψ from A to B/J_n for some sufficiently large n with $\phi = \pi \circ \psi$ (where $\pi : B/J_n \to B/J$ is the quotient map). Such a ψ is called a *G-equivariant partial lift* of ϕ.

When a C*-algebra A has a specified or understood G-action for which the corresponding C*-dynamical system is G-equivariantly semiprojective, we will sometimes, by slight abuse of terminology, say that A is equivariantly semiprojective.

We have written the definition for dynamical systems with general locally compact G, but actually we do not know a single example of an equivariantly semiprojective C*-dynamical system (A, G, α) with G noncompact. One possibility is $(\mathbb{C}, G, \iota)$, where G is a discrete Property T group and ι is the trivial action.

Equivariant semiprojectivity of a system (A, G, α) is sensitive to the group G in both directions. If H is a closed subgroup of G, there is no obvious implication either way between equivariant semiprojectivity of (A, G, α) and of $(A, H, \alpha|_H)$: on the one hand, if (B, G, β, J_n, ϕ) is a tower, it is harder to find a G-equivariant partial lift than an H-equivariant one; on the other hand, to prove equivariant semiprojectivity of $(A, H, \alpha|_H)$ more lifting problems must be considered. Taking the special case of H trivial, there is no obvious relation either way between equivariant semiprojectivity of (A, G, α) and ordinary semiprojectivity of A, and there does not appear to be any in general.

Similarly, if H is a closed normal subgroup of G, and α is a G-action on A which is trivial on H, and hence defines a (G/H)-action $\bar{\alpha}$ on A, equivariant semiprojectivity of (A, G, α) obviously implies equivariant semiprojectivity of $(A, G/H, \bar{\alpha})$ since a (G/H)-action may be regarded as a G-action, but the converse is not obvious (and is false in general) since to show equivariant semiprojectivity of (A, G, α) towers (B, G, β, J_n, ϕ) must be considered where $\beta|_H$ is not trivial (but only asymptotically trivial mod J_n on $\phi(A)$).

Equivariant Semiprojectivity of Trivial Actions

If G is a locally compact group and A a C*-algebra, denote the trivial action of G on A by ι.
The first result is obvious:

Proposition.

If (A, G, ι) is an equivariantly semiprojective C*-dynamical system, then A is a semiprojective C*-algebra.

Indeed, an ordinary lifting problem can be regarded as a G-equivariant lifting problem where all actions are trivial.

The converse is true if G is compact:

Proposition.

Let A be a semiprojective C*-algebra and G a compact group. Then (A, G, ι) is semiprojective.

This is easily proved by an averaging argument.

Semiprojectivity of A is, however, not sufficient for (A, G, ι) to be equivariantly semiprojective in general if G is noncompact, as the following example shows.

Example.

The system $(\mathbb{C}, \mathbb{Z}, \iota)$ is not $\mathbb{Z}$-equivariantly semiprojective.

Note that G-equivariant semiprojectivity of the system $(\mathbb{C}, G, \iota)$ is equivalent to the following:

Reformulation.

If (B, G, β) is a C*-dynamical system, (J_n) an increasing sequence of closed G-invariant ideals of B with $J = [\cup J_n]^-$, and q is a G-invariant projection in B/J, then for some sufficiently large n there is a G-invariant projection $p \in B/J_n$ with $\pi(p) = q$.

This fails for $G = \mathbb{Z}$ (and probably for most noncompact groups). Let u be the bilateral shift on $\mathcal{H} = \ell^2(\mathbb{Z})$. Then conjugation by u gives an automorphism of the compact operators $\mathbb{K} = \mathcal{K}(\mathcal{H})$. There are no nonzero finite-rank projections exactly commuting with u (to see this most easily, write operators as infinite matrices; the matrix of an operator commuting with u is constant on diagonals, but the matrix of a compact operator almost vanishes outside a finite block). However, for any $\epsilon > 0$ there are nonzero finite-rank projections approximately commuting with u within ϵ (e.g. any finite-rank subprojection of a spectral projection of u corresponding to a small neighborhood of 1).

Let B be the C*-algebra of bounded sequences from $\mathbb{K}$, with $\mathbb{Z}$-action given by coordinatewise conjugation by u. Let J_n be the ideal of sequences which are 0 after the n'th term, and $J = [\cup J_n]^-$ the ideal of sequences converging to 0. For each n let p_n be a nonzero finite-rank projection in $\mathbb{K}$ approximately commuting with u within $\frac{1}{n}$. Then the image q of the sequence (p_n) in B/J is invariant under the induced action of $\mathbb{Z}$; but it has no invariant projection preimage in B/J_n for any n.

The same construction works if G is any noncompact amenable group: let G act by translation (regular representation) on $\mathcal{H} = L^2(G)$, giving a G-action on $\mathcal{K}(\mathcal{H})$. Then there are almost invariant vectors, hence almost invariant one-dimensional projections, but no finite-dimensional invariant subspaces. Thus $(\mathbb{C}, G, \iota)$ is not G-equivariantly semiprojective.

Equivariant Semiprojectivity vs. Crossed Product Semiprojectivity

If (A, G, α) is a C*-dynamical system with α nontrivial, it is not at all clear that equivariant semiprojectivity of the system implies that A is a semiprojective C*-algebra, and indeed this seems doubtful, even if G is finite.

Equivariant semiprojectivity of (A, G, α) seems much more closely related to semiprojectivity of $A \rtimes_\alpha G$, although there is no implication either way in general.

Example.

Semiprojectivity of the crossed product does not imply equivariant semiprojectivity of the system, at least if the group is noncompact. Even semiprojectivity of the crossed product, semiprojectivity of $C^*(G)$, and conditional semiprojectivity of $A \rtimes_\alpha G$ with respect to $C^*(G)$ do not imply equivariant semiprojectivity of the system. The system $(\mathbb{C}, \mathbb{Z}, \iota)$ is a counterexample.

Example.

Equivariant semiprojectivity of the system does not imply semiprojectivity of the crossed product: the system $(\mathbb{C}, \mathbb{T}, \iota)$ is a counterexample (note that $C^*(\mathbb{T}) \cong C_0(\mathbb{Z})$ is not semiprojective).

However, we have the following result:

Theorem.

Let (A, G, α) be a C*-dynamical system. If the system is G-equivariantly semiprojective, and $C^*(G)$ is semiprojective, then $A \rtimes_\alpha G$ is semiprojective.

Not many groups appear to have semiprojective group C*-algebras, but finite groups, $\mathbb{Z}$, $\mathbb{R}$, and free groups on finitely many generators are a few examples. No infinite compact group has a semiprojective group C*-algebra (the group C*-algebra of a compact group is an AF algebra).

The Finite-Dimensional Case

Theorem.

(Phillips) Let (A, G, α) be a C*-dynamical system, with A finite-dimensional and G finite. Then (A, G, α) is equivariantly semiprojective.

This result would appear to be routine considering the ease of the non-equivariant version. But all evidence is that the equivariant case is far more subtle.

Let $G = D_6 \cong S_3$ be the dihedral group of order 6. G has the following presentation:

$$G = \langle g, h | h^3 = 1, g^2 = 1, ghg = h^{-1} \rangle \ .$$

Represent G on $\mathbb{C}^2$ by

$$h \leftrightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \ , \quad g \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where λ is a primitive cube root of unity. The representation is irreducible. The group generated by these unitaries is isomorphic to G and gives an inner action α on $\mathbb{M}_2$ by conjugation.

Theorem.

The C*-dynamical system $(\mathbb{M}_2, D_6, \alpha)$ is equivariantly semiprojective.

Let (B, G, β) be a C*-dynamical system with $G = D_6$, (J_n) an increasing sequence of invariant ideals of B, and $J = [\cup J_n]^-$, and $\phi : \mathbb{M}_2 \to B/J$ an equivariant homomorphism. We may assume that B and ϕ are unital.

If $\{e_{ij}\}$ are the standard matrix units in $\mathbb{M}_2$, we have

$$\alpha_h(e_{11}) = e_{11} \ , \ \ \alpha_h(e_{22}) = e_{22} \ , \ \ \alpha_h(e_{12}) = \bar{\lambda} e_{12}$$

$$\alpha_g(e_{11}) = e_{22} \ , \ \ \alpha_g(e_{22}) = e_{11} \ , \ \ \alpha_g(e_{12}) = e_{21}$$

so we must lift the e_{ij} to a set of matrix units transforming similarly under β.

Notation: write $q = e_{11}$, and $v = e_{12} + e_{21}$; then v is the self-adjoint unitary implementing α_g, $\lambda q + \bar{\lambda}(1-q)$ the unitary implementing α_h, and $e_{22} = 1 - q$, $e_{12} = qv(1-q)$, $e_{21} = (1-q)vq$. We will write the matrix units as

$$\{q, 1-q, qv(1-q), (1-q)vq\} \ .$$

We have $\alpha_h(q) = q$, $\alpha_g(q) = 1 - q$, $\alpha_h(qv(1-q)) = \lambda^2 qv(1-q)$, and $\alpha_g(qv(1-q)) = (1-q)vq = [qv(1-q)]^*$.

We must partially lift q and v to satisfy these relations for β. This can be done with care using functional calculus.

We first partially lift q to a projection $p \in B/J_n$ for some n with the properties that $\beta_h(p) = p$ and $\beta_g(p) = 1 - p$. We show this can be done. First note that if H is the subgroup of G generated by h, then H is normal and $p \in (B/J)^H$. Thus, replacing B by B^H and β by the induced action, we can work entirely in subalgebras on which β_h is trivial. Partially lift q to a projection $p' \in (B/J_n)^H$ for some n. By increasing n if necessary, we may make

$$\|\beta_g(p') - (1 - p')\|$$

arbitrarily small, and thus $z = p' - \beta_g(p')$ (which is self-adjoint) arbitrarily close to a unitary, i.e. $\|z^2 - 1\|$ arbitrarily small (< 1 suffices). Then z is invertible, $\beta_g(z) = -z$, and $\beta_g(z^2) = z^2$, so $\beta_g((z^2)^{-1/2}) = (z^2)^{-1/2}$ and $w = z(z^2)^{-1/2}$ is a self-adjoint unitary with $\beta_g(w) = -w$ (and $\beta_h(w) = w$ since everything is done within $(B/J_n)^H$). Set $p = (w + 1)/2$; then p is the desired lift. (Note that we have $\pi(z) = \pi(w) = q - (1 - q)$, so $\pi(p) = q$.)

It remains to equivariantly partially lift $e_{12} = qv(1-q)$. First, increasing n if necessary, partially lift v to a self-adjoint unitary u with $\beta_g(u) = u$ (this can be done by semiprojectivity of $C^*(v)$). Set $x = pu(1-p)$. We have $\beta_g(x) = x^*$. By further increasing n if necessary, we may arrange that $\|x^*x - (1-p)\|$, $\|xx^* - p\|$, and $\|\beta_h(x) - \lambda^2 x\|$ are arbitrarily small (since they are all zero mod J). Set

$$y = \frac{1}{3}\sum_{k=0}^{2} \bar{\lambda}^{2k}\beta_{h^k}(x) = p\left[\frac{1}{3}\sum_{k=0}^{2} \bar{\lambda}^{2k}\beta_{h^k}(u)\right](1-p)\ .$$

We then have $\beta_h(y) = \lambda^2 y$ and $\beta_g(y) = y^*$. If n is sufficiently large, we have $\|y - x\|$ is arbitrarily small, and hence $\|y^*y - (1-p)\|$ and $\|yy^* - p\|$ can be made arbitrarily small (< 1 suffices). We have $y^*y \leq 1-p$ and $yy^* \leq p$, so $y^*y \perp yy^*$, and $\beta_h(y^*y) = y^*y$, $\beta_h(yy^*) = yy^*$, $\beta_g(y^*y) = yy^*$, $\beta_g(yy^*) = y^*y$.

Set $a = y^*y + yy^*$. We then have $\beta_g(a) = \beta_h(a) = a$, and $(yy^*)y = ay = ya = y(y^*y)$. If $\|y^*y - (1-p)\|$ and $\|yy^* - p\|$ are small, a is invertible, and $s = a^{-1/2}y = ya^{-1/2}$ is a partial isometry with $s^*s = 1 - p$, $ss^* = p$, $\beta_g(s) = s^*$, and $\beta_h(s) = \lambda^2 s$. We have $\pi(x) = \pi(y) = qv(1-q)$, so $\pi(a) = 1$ and $\pi(s) = qv(1-q)$ and s is the desired equivariant partial lift of $qv(1-q)$. Then s^* is an equivariant partial lift of $(1-q)vq$ and we are finished.