Hilbert Spaces Without Countable AC

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We are accustomed to assume the Axiom of Choice. But what if we don't? Things can become strange and counterintuitive, but (arguably) more interesting.

We study Hilbert spaces and operators without any form of Choice, especially Hilbert spaces whose dimension is "finite" in a sense incompatible with even the Countable AC.

Axiom of Choice (AC):

If $\{X_i : i \in I\}$ is a collection of nonempty sets, there is a *choice function* $c : I \to \cup_i X_i$ with $c(i) \in X_i$ for all i.

Equivalently: if $\{X_i : i \in I\}$ is a collection of nonempty sets, then $\prod_{i \in I} X_i$ is nonempty. (*Multiplicative Axiom*)

Countable Axiom of Choice (CC):

If $\{X_n : n \in \mathbb{N}\}$ is a sequence of nonempty sets, there is a *choice function* $c : \mathbb{N} \to \cup_n X_n$ with $c(n) \in X_n$ for all n.

There are many other versions of Choice: Dependent Choice, BPI, ...

Without AC we cannot use many of the usual tools:

- Zorn's Lemma
- Tikhonov's Theorem
- Hahn-Banach Theorem
- Baire Category Theorem

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which all require some form of Choice.

Without CC we even cannot prove

- Sequential criteria for continuity and closure
- A countable union of countable sets is countable
- ...

But we can still make finitely many choices.

Thus we must set aside almost everything we think we know about Hilbert spaces, operators, and C*-algebras, and prove or reprove everything from scratch.

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Some things work identically with the same proofs

Some things are still true but require different proofs

Some things are no longer true.

Dedekind-finite Sets

Definition:

A set X is *Dedekind-finite* if every injective function from X to X is surjective. Otherwise X is *Dedekind-infinite*. A cardinal is Dedekind-finite if it is the cardinal of a Dedekind-finite set.

Any finite set is Dedekind-finite. Any subset of a Dedekind-finite set is Dedekind-finite.

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Proposition:

A set X is Dedekind-infinite if and only if it has a countably infinite subset (a sequence of distinct elements).

Under CC, every infinite set is Dedekind-infinite, i.e. there are no DF sets.

If we have one DF cardinal κ, we have many: the sequence $(\kappa - n)$ is a strictly decreasing sequence of DF cardinals, contradicting the Well-ordering Principle. If X is DF and Y is a proper subset of X, then $|Y| < |X|$. More bizarrely:

There is a set of DF cardinals order-isomorphic to $\mathbb{R}$.

In some models of ZF, there are $2^{\aleph_0}$ mutually incomparable DF cardinals.

In some models of ZF, the collection of DF cardinals is not a set: every set is an image of a DF set.

Stronger Flavors

Notation: $|X|$ is the cardinality of X.
$\mathcal{P}(X)$ (power set of X) is the set of all subsets of X.
$Fin(X)$ is the set of all finite subsets of X.

We have $|X| \leq |Fin(X)| \leq |\mathcal{P}(X)|$.

Definition:

Let X be a set.

(i) X is *Cohen-finite* if $Fin(X)$ is Dedekind-finite.
(ii) X is *power Dedekind-finite* if $\mathcal{P}(X)$ is Dedekind-finite.

CF = "infinite and Cohen-finite."
PF = "infinite and power Dedekind-finite."

X PF $\Rightarrow$ X CF $\Rightarrow$ X DF. No reverse implications.

Proposition:

If X is an infinite set, then $\mathcal{P}(Fin(X))$ (and hence $\mathcal{P}(\mathcal{P}(X))$) is Dedekind-infinite.

X is PF if and only if X does not have a sequence of distinct subsets, and is CF if and only if it does not have a sequence of distinct finite subsets.

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Theorem (of ZF!):

Let X be a set. If X has a sequence of distinct subsets, then X has a sequence of pairwise disjoint nonempty subsets. (The converse is trivial.)

Corollary:

Let X be an infinite set. Then X is PF if and only if X does not have a sequence of pairwise disjoint nonempty subsets. X is CF if and only if it does not have a sequence of pairwise disjoint nonempty finite subsets.

Russell Socks

Definition:

A set X is a *Russell set*, or a *set of Russell socks*, if it is DF and is a union of a sequence of pairwise disjoint two-element sets.

The two-element subsets are "pairs of socks" where there is no way to globally choose a left or right sock out of each pair.

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If you have a set of Russell socks and add another pair of socks, you have *more* socks (strictly larger cardinality), but the *same* number ($\aleph_0$) of pairs of socks.

Amorphous Sets

Definition:

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An amorphous set is PF. Any infinite subset of an amorphous set is amorphous.

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An amorphous set is PF. Any infinite subset of an amorphous set is amorphous.

Proposition:

An amorphous set cannot be totally ordered.

Rigid Sets

Definition:

An infinite set X is *rigid* if every permutation of X moves only finitely many elements of X. X is *strongly rigid* if, whenever X is partitioned into nonempty subsets, all but finitely many of the subsets are singletons.

strongly rigid $\Rightarrow$ rigid $\Rightarrow$ DF and strongly rigid $\Rightarrow$ PF. Any subset of a [strongly] rigid set is [strongly] rigid.
strongly rigid does not imply amorphous (or conversely).

Definition:

A set X is *strongly amorphous* if it is amorphous and strongly rigid.

Existence and Models

Fraenkel (1922): There is a model of ZF + atoms with a strongly amorphous set.

Cohen (1963): There is a model of ZF in which $\mathbb{R}$ has a CF subset.

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These models can be combined via transfer theorems to give a single model of ZF containing all flavors of DF sets.

Completeness

Without CC, the usual notion of completeness (*Cauchy completeness*) for metric spaces is not appropriate. We instead use:

Definition:

A metric space (X, ρ) is *σ-complete* if, whenever (A_n) is a decreasing sequence of closed subsets of X with $diam(A_n) \to 0$, $\cap_n A_n \neq \emptyset$ (it is necessarily a singleton).

Theorem:

Let (X, ρ) be a metric space. The following are equivalent:

(i) (X, ρ) is σ-complete.

(ii) X is complete in the uniform structure from ρ.

(iii) (X, ρ) is *absolutely closed*: if ϕ is an isometry from (X, ρ) to a metric space (Y, τ), then $\phi(X)$ is closed.

These imply

(iv) (X, ρ) is Cauchy complete.

If the CC is assumed, (i)–(iv) are all equivalent.

Hilbert Spaces

Definition:

A *Hilbert space* is a complex inner product space which is σ-complete.

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Most important property:

Theorem [Closest Vector Property]:

Let $\mathcal{H}$ be a Hilbert space and C a nonempty closed convex subset of $\mathcal{H}$. For any $\xi \in \mathcal{H}$, there is a unique $\eta \in C$ such that

$$\|\xi - \eta\| = \rho(\xi, \eta) = \rho(\xi, C) = \inf_{\zeta \in C} \|\xi - \zeta\| = \min_{\zeta \in C} \|\xi - \zeta\| .$$

The usual proof uses Cauchy sequences, with a lemma from the Parallelogram Law.

There is an alternate proof not using CC:

Proof: Let $r = \rho(\xi, C)$. For each $n \in \mathbb{N}$ let A_n be the intersection of C with the closed ball of radius $r + \frac{1}{n}$ around ξ. Then each A_n is a nonempty closed convex set in $\mathcal{H}$, with $A_{n+1} \subseteq A_n$, and $\rho(\xi, A_n) = \rho(\xi, C)$. Then $diam(A_n) \to 0$ as $n \to \infty$. Thus by σ-completeness $\cap_n A_n = \{\eta\}$ for some $\eta \in C$. Clearly $\|\xi - \eta\| = r$ and η is the unique vector in C with this property.

Other crucial properties of Hilbert spaces follow in the usual way:

1. If $\mathcal{H}$ is a Hilbert space and $\mathcal{Y}$ a closed subspace, then $\mathcal{Y}$ has an orthogonal complement, and there is an orthogonal projection $P_{\mathcal{Y}}$ from $\mathcal{H}$ onto $\mathcal{Y}$.

2. [Riesz Representation Theorem] If ϕ is a bounded linear functional on a Hilbert space $\mathcal{H}$, then there is a unique vector $\eta \in \mathcal{H}$ with $\phi(\xi) = \langle \xi, \eta \rangle$ for all $\xi \in \mathcal{H}$.

3. Every bounded operator on a Hilbert space has an adjoint.

All these can fail if the inner product space is only Cauchy complete.

Orthonormal Bases

Under AC, we have the standard result:

Theorem:

Let $\mathcal{H}$ be a Hilbert space. Then

(i) $\mathcal{H}$ has an orthonormal basis.

(ii) Every orthonormal set in $\mathcal{H}$ can be expanded to an orthonormal basis.

(iii) Any two orthonormal bases for $\mathcal{H}$ have the same cardinality (orthogonal dimension is well defined).

No Choice is needed to prove this for separable Hilbert spaces (Gram-Schmidt).

All parts of this theorem can fail if the AC is not assumed. Thus Hilbert spaces are much more varied and interesting without Choice. There are even infinite-dimensional Hilbert spaces which are so different that every bounded operator between them has finite rank.

The Hilbert spaces with orthonormal bases can be nicely described as ℓ^2 spaces.

Definition:

Let X be a set. Then

$$\ell^2(X) = \left\{ \eta : X \to \mathbb{C} : \sum_{x \in X} |\eta(x)|^2 < \infty \right\} .$$

Here "sum over X" means "integral with respect to counting measure on X." Thus the sum of a nonnegative function over X is the supremum of the sums over finite subsets of X.

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There is a natural inner product on $\ell^2(X)$ defined in the obvious way.

Proposition:

If X is a set, then $\ell^2(X)$ is σ-complete, hence a Hilbert space.

There is a natural orthonormal basis $\{\xi_x : x \in X\}$ for $\ell^2(X)$, where ξ_x is the characteristic (indicator) function of $\{x\}$.

Conversely, we have:

Proposition:

Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{\eta_x : x \in X\}$ indexed by a set X. then $\eta_x \mapsto \xi_x$ extends to an isometric isomorphism $\mathcal{H} \cong \ell^2(X)$.

Thus the spaces $\ell^2(X)$ are universal models for Hilbert spaces with orthonormal bases.

We will be particularly interested in $\ell^2(X)$ for DF X. There are some surprises.

Note first that if X is DF, then $\ell^2(X)$ is not separable (the basis vectors cannot all be closely approximated by a countable set).

In the non-Choice setting, a nonseparable Hilbert space is not necessarily "larger" or "smaller" than a separable one (just "different"!)

Russell Socks

The first interesting example is to take X to be a set of Russell socks, $X = \cup_n X_n$ where X_n has two elements, and X is DF.

Let $\mathcal{X}_n$ be the span of X_n in $\ell^2(X)$. The $\mathcal{X}_n$ are mutually orthogonal two-dimensional subspaces, each with a distinguished (unordered!) orthonormal basis.
For each n let η_n be the normalized sum of the two basis vectors in $\mathcal{X}_n$. Then the η_n are an orthonormal sequence in $\ell^2(X)$. Let $\mathcal{Y}$ be the closed span of the η_n. Then $\mathcal{Y} \cong \ell^2(\mathbb{N})$. Set $\mathcal{Z} = \mathcal{Y}^\perp$.

Thus $\ell^2(X)$ for DF X can contain an orthonormal sequence! Even more is true:

$$\begin{aligned}\ell^2(X \sqcup \mathbb{N}) &\cong \ell^2(\mathbb{N}) \oplus \ell^2(X) \cong \ell^2(\mathbb{N}) \oplus \mathcal{Y} \oplus \mathcal{Z} \\ &\cong \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \oplus \mathcal{Z} \cong \ell^2(\mathbb{N}) \oplus \mathcal{Z} \cong \mathcal{Y} \oplus \mathcal{Z} \cong \ell^2(X)\end{aligned}$$

So $\ell^2(X)$ has an orthonormal basis indexed by the DF set X, and another orthonormal basis indexed by the Dedekind-infinite set $X \sqcup \mathbb{N}$. Thus orthogonal dimension is not well defined in general.

The subspaces $\mathcal{Y}$ and $\mathcal{Z}$ can be viewed another way. There is a permutation of X which interchanges the two socks in each pair, which defines a self-adjoint unitary operator U on $\ell^2(X)$. $\mathcal{Y}$ and $\mathcal{Z}$ are the $+1$ and -1 eigenspaces of U.

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$\mathcal{Z}$ is spanned by the differences of the basis vectors in each X_n. Thus $\mathcal{Z}$ is the closed span of a sequence of one-dimensional subspaces of $\ell^2(X)$. However, the difference of the basis vectors in $\mathcal{X}_n$ is only well defined up to sign, and a global choice of sign is not possible. Thus an orthonormal sequence cannot be made from these vectors. ($\mathcal{Z}$ is in fact not separable.)

It seems plausible that $\mathscr{Z}$ does not have an orthonormal basis. (This is claimed by N. Brunner *et al.*, but we cannot follow the alleged proof.)

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But using a set of "super Russell socks," with analogous U, $\mathcal{Y}$, $\mathcal{Z}$, we can show:

Theorem:

There is (in a model of ZF) an amorphous set X, and an infinite-dimensional closed subspace $\mathcal{Z}$ of $\ell^2(X)$, such that every orthonormal set in $\mathcal{Z}$ is finite. In particular, $\mathcal{Z}$ has no orthonormal basis.

This $\mathcal{Z}$ is of course a closed subspace of a Hilbert space with an orthonormal basis. We do not know whether every Hilbert space is isomorphic to a subspace of a Hilbert space with an orthonormal basis.

CF Sets

If X is CF, the Hilbert space $\ell^2(X)$ has some curious properties:

Proposition:

Let X be a CF set. Then

(i) Every vector in $\ell^2(X)$ has finite support (i.e. the orthonormal basis $\{\xi_x : x \in X\}$ is a Hamel basis).

(ii) Every sequence of vectors in $\ell^2(X)$ has finite common support.

(iii) $\ell^2(X)$ does not contain an orthonormal sequence.

As a result, the closed unit ball of $\ell^2(X)$ is a σ-complete metric space which is sequentially compact but not compact (not totally bounded).

Here is an interesting example. Let X be a CF set which is not PF, and write $X = \sqcup_n X_n$ with nonempty X_n. Let $T \in \mathcal{B}(\ell^2(X))$ be $\frac{1}{n}1$ on the closed span $\mathcal{X}_n$ of X_n. Then T is a bijection, but T^{-1} is not bounded. T and T^{-1} are self-adjoint.

This gives a simple counterexample to the Open Mapping Theorem and Closed Graph Theorem. Truncations of T^{-1} give a counterexample to the Uniform Boundedness Theorem.

The usual proofs of these theorems use some version of the Baire Category Theorem. The theorems are actually essentially equivalent to CC.

Bounded Operators

If $\mathcal{H}$ is a Hilbert space, many basic properties of $\mathcal{B}(\mathcal{H})$ persist without AC: support projections, polar decomposition, continuous and Borel functional calculus of self-adjoint elements, ...

But there can be dramatically different behavior for $\mathcal{H} = \ell^2(X)$, X sufficiently DF:

Theorem:

Let X be CF, and $T \in \mathcal{B}(\ell^2(X))$. Then X decomposes into finite subsets $\{X_i : i \in I\}$ such that the span $\mathcal{X}_i$ of X_i is invariant under T for each i.

In particular, T has many finite-dimensional invariant subspaces.

Corollary:

If X is CF, then $\mathcal{B}(\ell^2(X))$ is stably finite.

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Let X be a strongly amorphous set. Then every bounded operator on X is a finite-rank perturbation of a scalar.

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This has an interesting reformulation:

Corollary:

Let X be a strongly amorphous set. Then $\mathcal{H} = \ell^2(X)$ is infinite-dimensional, and every closed subspace of $\mathcal{H}$ has either finite dimension or finite codimension.

Thus $\mathcal{H}$ is a "hereditarily indecomposable Hilbert space." We suspect "closed" can be eliminated from the statement.

Spectrum

The spectrum of an operator (or element of any unital Banach algebra) has the usual properties.

Theorem:

If X is PF and $T \in \mathcal{B}(\ell^2(X))$, then $\sigma(T)$ is finite, and every $\lambda \in \sigma(T)$ is an eigenvalue.

Spectrum

The spectrum of an operator (or element of any unital Banach algebra) has the usual properties.

Theorem:

If X is PF and $T \in \mathcal{B}(\ell^2(X))$, then $\sigma(T)$ is finite, and every $\lambda \in \sigma(T)$ is an eigenvalue.

On the other hand, if X is not power Dedekind-finite, then every separable compact subset of $\mathbb{C}$ occurs as the spectrum of a normal operator on $\ell^2(X)$.

Compact Operators

The following definition of compact operator seems to be the best one (with or without AC):

Definition:

An operator on a Hilbert space is *compact* if the image of every bounded set is totally bounded. Write $\mathcal{K}(\mathcal{H})$ for the set of compact operators on the Hilbert space $\mathcal{H}$.

Proposition:

Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{K}(\mathcal{H})$ is a closed two-sided ideal in $\mathcal{B}(\mathcal{H})$, and is the closure of the ideal of finite-rank operators.

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The Calkin Algebra

If $\mathcal{H}$ is an infinite-dimensional Hilbert space, we define the *Calkin algebra* $\mathcal{Q}(\mathcal{H})$ to be the quotient C*-algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

If $\mathcal{H}$ is infinite-dimensional and separable, then $\mathcal{Q}(\mathcal{H})$ is simple and purely infinite. But for other Hilbert spaces it can be very different.

Theorem:

If $\mathcal{H}$ is an infinite-dimensional Hilbert space, then the following are equivalent:

(i) $\mathcal{B}(\mathcal{H})$ is finite.

(ii) $\mathcal{B}(\mathcal{H})$ is stably finite.

(iii) $\mathcal{Q}(\mathcal{H})$ is finite.

(iv) $\mathcal{Q}(\mathcal{H})$ is stably finite.

(v) $\mathcal{H}$ does not contain an orthonormal sequence.

If $\mathcal{H} = \ell^2(X)$ for a set X, these are equivalent to

(vi) X is CF.

If X is strongly amorphous, then $\mathcal{Q}(\ell^2(X)) \cong \mathbb{C}$.

If $Y = X \times \{0, \ldots, n-1\}$, then $\mathcal{Q}(\ell^2(Y)) \cong \mathbb{M}_n$.

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If $Y = X \times \{0, \dots, n-1\}$, then $\mathcal{Q}(\ell^2(Y)) \cong \mathbb{M}_n$.

$\mathcal{Q}(\mathcal{H})$ is not necessarily simple: if X is strongly amorphous, and $\mathcal{H} = \ell^2(X) \oplus \ell^2(\mathbb{N}) \cong \ell^2(X \sqcup \mathbb{N})$, then $\mathcal{Q}(\mathcal{H}) \cong \mathcal{Q}(\ell^2(X)) \oplus \mathcal{Q}(\ell^2(\mathbb{N}) \cong \mathbb{C} \oplus \mathbb{Q}(\ell^2(\mathbb{N}))$.

We have an example of an X such that $\mathcal{Q}(\ell^2(X))$ is commutative and nonseparable (isomorphic to $\ell^\infty(Y)/c_0(Y)$ for an infinite set Y).

We suspect $\mathcal{Q}(\mathcal{H})$ is rarely if ever separable if infinite-dimensional.

We also suspect $\mathcal{Q}(\mathcal{H})$ is rarely simple if $\mathcal{H}$ is nonseparable. The ideal structure of $\mathcal{Q}(\ell^2(X))$ should be related to the poset of infinite cardinals infinitely dominated by $|X|$.

If $\mathcal{H}$ has no orthonormal basis, we have no clear idea of the structure of $\mathcal{Q}(\mathcal{H})$.

New Cardinal Flavors and Relations

Definition:

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space.

(i) $\mathcal{H}$ is *HDF* if it is not isometric to a proper subspace of itself.

(ii) $\mathcal{H}$ is *HCF* if it does not have a sequence of mutually orthogonal nonzero finite-dimensional subspaces.

(iii) $\mathcal{H}$ is *HPF* if it does not have a sequence of mutually orthogonal nonzero (closed) subspaces.

(iv) $\mathcal{H}$ is *Hilbert-amorphous* if every closed subspace has either finite dimension or finite codimension.

Hilbert-amorphous $\Rightarrow$ HPF $\Rightarrow$ HCF $\Rightarrow$ HDF. The implications are not reversible.

$\mathcal{H}$ is HDF if and only if it does not contain an orthonormal sequence.

Corresponding notions for sets: if X is a set, then X is HDF (etc.) if $\ell^2(X)$ is HDF (etc.)

We have X HDF $\Leftrightarrow$ X CF.

We have X HCF $\Rightarrow$ X CF, X HPF $\Rightarrow$ X PF, X Hilbert-amorphous $\Rightarrow$ X amorphous. X strongly amorphous $\Rightarrow$ X Hilbert-amorphous.

The converses seem doubtful. Thus HCF, HPF, and Hilbert-amorphous are potentially new flavors of Dedekind-finiteness not considered by set theorists.

We also have a potentially new preorder and equivalence relation on cardinals:

Definition:

Let X and Y be sets. Then $|Y| \preceq |X|$ if $\ell^2(X)$ contains a subspace isometric to $\ell^2(Y)$ (i.e. $\ell^2(X)$ contains an orthonormal set of cardinality $|Y|$.
$|X| \sim |Y|$ if $\ell^2(X) \cong \ell^2(Y)$.

There is a Schröder-Bernstein Theorem: $|X| \sim |Y|$ if and only if $|X| \preceq |Y|$ and $|Y| \preceq |X|$.

The usual order $\leq$ is much stronger than $\preceq$: if X is a set of Russell socks, then $\aleph_0 \preceq |X|$, and $|X| \sim |X \sqcup \mathbb{N}|$.

Project: Compute the K-theory and nonstable K-theory of $\mathcal{B}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})$ for Hilbert spaces of the kind we have been considering.

The answer will depend on what model of ZF we are working in.

C*-Algebras

We use the usual definition of C*-algebra, but require σ-completeness. Many basic properties of C*-algebras continue to hold.

Examples: (i) $\mathcal{B}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$.
(ii) $C_0(X)$ for any locally compact Hausdorff space X.

Under AC, these examples are universal by the two Gelfand-Naimark theorems.

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Both Gelfand-Naimark theorems can fail without AC.

Representability

Definition:

A C*-algebra A is *representable* if it is (isometrically) *-isomorphic to a norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

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Definition:

A C*-algebra A *has enough states* if

$$\|x\|^2 = \sup\{\phi(x^*x) : \phi \in \mathcal{S}(A)\}$$

for every $x \in A$.

The Hahn-Banach Theorem is needed to give enough states.

Proposition:

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Example: If the Axiom of Determinacy is assumed, the commutative C*-algebra $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ has *no* states! (Thus in particular is not representable.)

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Example: If the Axiom of Determinacy is assumed, the commutative C*-algebra $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ has *no* states! (Thus in particular is not representable.)

The Hahn-Banach Theorem can be proved for separable Banach spaces without any Choice. Thus every separable C*-algebra is representable, and is in fact representable on a separable Hilbert space.

Commutative C*-Algebras and Functional Calculus

Gelfand's Theorem is dicier, even for separable commutative C*-algebras: need enough pure states (maximal ideals) + more. In the nonseparable case, $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ is a counterexample.

Commutative C*-Algebras and Functional Calculus

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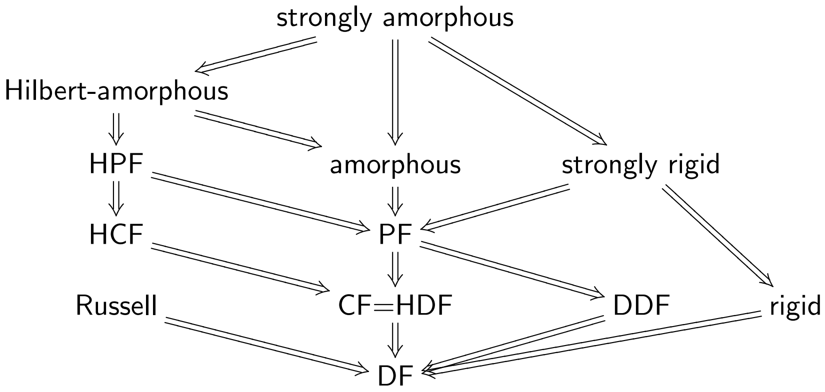
We have a proof in the separable case, but it is very roundabout: using model theory, it can be shown that the statement does not depend on the AC (i.e. if there is a ZFC proof, there is a ZF proof). Since there is a ZFC proof, there is a ZF proof.


strongly amorphous
Hilbert-amorphous
HPF
amorphous
strongly rigid
HCF
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Russell
CF=HDF
DDF
rigid
DF