CLASSIFICATION OF C*-ALGEBRAS

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Questions to be answered:

1. What is a C*-algebra?

2. What is a classification?

3. How do we classify C*-algebras?

4. Why do we care?

1. What is a C*-algebra?

Definition. A *concrete C*-algebra (of operators)* is a norm-closed *-subalgebra of $\mathcal{L}(\mathcal{H})$, for a Hilbert space $\mathcal{H}$.

Definition. An *(abstract) C*-algebra* is a Banach *-algebra satisfying the C*-axiom.

Banach algebra: (Complex) Banach space which is an algebra over $\mathbb{C}$, satisfying

$$\|xy\| \leq \|x\|\|y\|$$

Involution: $* : A \to A$, $(x + y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$, $(xy)^* = y^* x^*$, $(x^*)^* = x$

C*-axiom: $\|x^* x\| = \|x\|^2$

Theorem. (Gelfand-Naimark-Segal) Every (abstract) C*-algebra is isometrically isomorphic to a concrete C*-algebra of operators.

Examples of C*-Algebras

$\mathbb{M}_n$, the $n \times n$ matrices over C $(= \mathcal{L}(\mathbb{C}^n))$

$\mathbb{K}$, the compact operators on $\mathcal{H}$

$C_0(X)$, X a locally compact Hausdorff space

$C_0(X, \mathbb{M}_n)$ and $C_0(X, \mathbb{K})$ (typical homogeneous C*-algebra)

$\{f \in C([0,1], \mathbb{M}_2) : f(1) \text{ diagonal}\}$ (typical subhomogeneous C*-algebra)

Type I C*-algebras: C*-algebras which locally "look like" C*-algebras like these (assembled from these by extensions, etc.)

Type I C*-algebras are "almost commutative."

Opposite extreme: simple C*-algebras (no nontrivial closed ideals)

General C*-algebras are "assembled" from simple C*-algebras

Early History (pre-1970's): C*-algebras closely tied to operator theory on Hilbert spaces.

Mackey Philosophy: To understand a C*-algebra, must understand its representation theory. (Possible only with *Type I* C*-algebras)

Modern Philosophy: C*-algebras can be studied and understood independent of their representation theory. (Non-type-I C*-algebras can be understood!)

Noncommutative Topology: C*-algebras are noncommutative generalizations of topological spaces, and can be studied by topological methods.

Applications:

1. Mathematical physics

quantum mechanics

2. Representation theory of locally compact topological groups

$G \mapsto C^*(G)$

3. Singular spaces

dynamical systems

foliations

locally compact groupoids

...

Space $\mapsto$ C*-algebra

Von Neumann Algebras

Definition. A *von Neumann Algebra* is a weakly closed unital *-algebra of operators on a Hilbert space.

Commutant: If $M \subseteq \mathcal{L}(\mathcal{H})$,

$$M' = \{x \in \mathcal{L}(\mathcal{H}) : xy = yx \ \forall y \in M\}$$

$M'' = (M')'$. $M \subseteq M''$, $(M'')' = M'$.

Bicommutant Theorem. M is a von Neumann algebra if and only if $M = M''$.

M is a *factor* if $\mathcal{Z}(M) = M \cap M'$ is $\mathbb{C}1$.

Every von Neumann algebra is a "continuous (measurable) direct sum" (direct integral) of factors.

The theory of von Neumann algebras is "noncommutative measure theory."

2. What is a classification?

A *true classification* consists of

(1) A "list" of standard objects in the class

(2) An "algorithm" for determining, for any given object, which standard object it is isomorphic to.

Examples:

Real vector spaces

closed 2-manifolds

finite simple groups

Bernoulli shifts (entropy)

countable abelian torsion groups (Ulm invariants)

Non-examples: knots, closed 3-manifolds

finitely presented groups

A *"classification"* is a complete description in terms of invariants or objects of a different type (e.g. a faithful functor to a different category).

Examples: knots via fundamental group of complement

commutative C*-algebras:

Theorem. (Gelfand-Naimark) If A is a commutative C*-algebra, then there is a unique locally compact Hausdorff space X such that A is isomorphic to $C_0(X)$. $X \mapsto C_0(X)$ is a contravariant category equivalence.

Classification theorems for C*-algebras are mostly "classifications" rather than true classifications.

They are nonetheless very useful in proving isomorphism theorems, and also in giving the existence of examples.

A *partial classification* is an incomplete description in terms of invariants.

Invariant not complete (or shown to be complete)

Classification restricted to subclass satisfying technical conditions

Example: Spaces via (co)homological invariants

Classification of Stable Homogeneous C*-Algebras

A C*-algebra is *stable homogeneous* if it arises as a locally trivial bundle over a locally compact Hausdorff space X, with fibers isomorphic to $\mathbb{K}$.

Can also define nonstable homogeneous C*-algebras. Finite homogeneous C*-algebras: locally trivial $\mathbb{M}_n$-bundles.

Canonical example: $C_0(X, \mathbb{K})$, $C_0(X, \mathbb{M}_n)$

Not every homogeneous C*-algebra is globally of this form: there is a global twist in general.

The transition functions from local trivializations define a 2-cocycle on X with values in the sheaf $\mathcal{U}$ of germs of continuous functions from X to $\mathbb{T}$.

The class of this cocycle in $H^2(X, \mathcal{U}) \cong H^3(X, \mathbb{Z})$ is the *Dixmier-Douady invariant* of A.

Theorem. (Dixmier-Douady) The Dixmier-Douady invariant is a complete isomorphism invariant for stable homogeneous C*-algebras. Every element of the group $H^3(X, \mathbb{Z})$ occurs.

The classification of nonstable homogeneous C*-algebras is more complicated, but can be done using topological invariants.

Type Classification of Factors

Definition A *projection* in a C*-algebra is an element p with $p = p^* = p^2$.

Projections in a concrete C*-algebra are (orthogonal) projection operators.

Von Neumann algebras contain "many" projections (spectral theorem)

Partial order: $p \leq q$ if $pq = qp = p$. p is a *subprojection* of q.

Definition. Projections $p, q \in A$ are *equivalent* (in A) if $\exists u \in A$ with $u^*u = p$, $uu^* = q$ (*partial isometry from* p *to* q). Write $p \sim q$.

If $p \sim q' \leq q$, p is *subordinate* to q. Write $p \precsim q$.

Definition. A projection is *finite* if it is not equivalent to a proper subprojection.

A is *finite* if 1_A is finite.

A factor may be finite (e.g. $\mathbb{M}_n$) or infinite (e.g. $\mathcal{L}(\mathcal{H})$, $\mathcal{H}$ infinite-dimensional)

A factor is *semifinite* if it contains nonzero finite projections.

A factor is *purely infinite*, or *Type III*, if every nonzero projection is infinite.

A factor is *discrete*, or *Type I*, if it contains minimal nonzero projections.

A Type I factor is isomorphic to $\mathcal{L}(\mathcal{H})$ for some $\mathcal{H}$. $\mathcal{L}(\mathcal{H})$ is *Type* I_n if $dim(\mathcal{H}) = n$.

A nondiscrete factor is *continuous*. A factor is *Type II* if it is semifinite and continuous.

Type II_1: finite and continuous

Type II_∞: infinite, semifinite, continuous

Type II_1, Type II_∞, and Type III factors exist (e.g. constructed from groups acting on measure spaces). There are uncountably many mutually nonisomorphic examples of each type on a separable Hilbert space.

Alternate description of type classification via dimension functions and traces.

Definition. A *dimension function* on a C*-algebra A is a function $D : Proj(A) \to [0, \infty]$ such that $D(p) = D(q)$ if $p \sim q$ and $D(p + q) = D(p) + D(q)$ if $p \perp q$ $(pq = 0)$.

It follows that $p \precsim q \Rightarrow D(p) \leq D(q)$.

Theorem. (Murray-von Neumann) Every factor M (on a separable Hilbert space) has a dimension function D, unique up to normalization, and $p \precsim q \Leftrightarrow D(p) \leq D(q)$. The range of D is

$\{0, 1, \ldots, n\}$ if M is Type I_n

$[0, 1]$ if M is Type II_1

$[0, \infty]$ if M is Type II_∞

$\{0, \infty\}$ if M is Type III.

Traces

Definition. A *trace* (tracial state) on a C*-algebra A is a positive linear functional τ of norm 1 such that $\tau(xy) = \tau(yx)$ for all $x, y \in A$.

Example: $\mathbb{M}_n$ has a unique trace, the "usual" one scaled so that $\tau(1) = 1$.

A trace restricted to projections gives a dimension function. Conversely:

Theorem. (Murray-von Neumann) The dimension function on a finite factor comes from a trace, i.e. a finite factor has a unique trace, which determines comparability of projections.

A Type I_∞ or Type II_∞ factor also has a "semifinite trace" (unbounded, not everywhere defined).

Classification of Injective Factors

Definition. A factor M on a separable Hilbert space is *approximately finite dimensional (afd)* (or *hyperfinite*) if any finite subset of M can be closely approximated in the *weak* topology by elements lying in a finite-dimensional C*-subalgebra (direct sum of matrix algebras.)

Theorem. (Murray-von Neumann) All hyperfinite II_1 factors are isomorphic.

Powers 1967: There are uncountably many mutually nonisomorphic afd factors of Type III.

Invariant is a number $\lambda \in [0, 1]$.

Tomita-Takesaki: Every von Neumann algebra has a "canonical" one-parameter group of automorphisms (σ_t) with (unbounded) positive generator Δ (*modular operator*)

Powers' invariant λ can be interpreted using the spectrum of Δ and $\{t : \sigma_t \text{ is inner }\}$.

Can use this invariant to divide Type III factors into Type III_λ, $0 \leq \lambda \leq 1$.

Theorem. (Connes) (i) There is a unique afd factor of Type II_∞.

(ii) The Powers factor is the unique afd factor of Type III_λ for $0 < \lambda < 1$.

(iii) There are uncountably many afd factors of Type III_0, "classified" by ergodic flows (Krieger).

There is also a unique afd factor of Type III_1 (Haagerup).

Connes also proved that the afd factors are precisely the *injective* factors (range of a conditional expectation from $\mathcal{L}(\mathcal{H})$.)

This result makes it much easier to identify when a factor is afd.

The afd factors are also the *amenable* factors.

Standard Operations on C*-Algebras:

extensions

Tensor product

Crossed product

Universal C*-algebras given by generators and relations

Inductive limit:

$$A_1 \xrightarrow{\phi_{12}} A_2 \xrightarrow{\phi_{23}} \quad \cdots \quad \longrightarrow A$$

A is denoted $\lim_{\rightarrow} A_n$ or $\lim_{\rightarrow}(A_n, \phi_{m,n})$.

Example: Let $\phi_{n,n+1} : \mathbb{M}_n \to \mathbb{M}_{n+1}$ be defined by

$$\phi_{n,n+1}(a) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

Then $\lim_{\rightarrow}(\mathbb{M}_n, \phi_{n,n+1}) \cong \mathbb{K}$.

Example: Let $A_n = \mathbb{M}_{2^n}$, $\phi_{n,n+1}(a) = diag(a, a)$.

$\lim_{\rightarrow}(A_n, \phi_{n,n+1})$ is called the *UHF algebra of type* 2^∞, denoted $\mathbb{M}_{2^\infty}$. (Also called the *CAR algebra*.)

Can also do for $A_n = \mathbb{M}_{3^n}$, $\phi_{n,n+1}(a) = diag(a, a, a)$. $\lim_{\rightarrow}(\mathbb{M}_{3^n}, \phi_{n,n+1}) = \mathbb{M}_{3^\infty}$.

Question: Are $\mathbb{M}_{2^\infty}$ and $\mathbb{M}_{3^\infty}$ isomorphic? If not, how do we distinguish them?

Idea: Use traces.

Since $\mathbb{M}_n$ has a unique trace, it follows that $\mathbb{M}_{2^\infty}$ and $\mathbb{M}_{3^\infty}$ also have unique traces.

Invariant: The set of values the trace takes on projections.

For $\mathbb{M}_n$, get $\left\{\frac{k}{n} : 0 \leq k \leq n\right\}$.

For $\mathbb{M}_{2^\infty}$, get the dyadic rationals in $[0, 1]$.

For $\mathbb{M}_{3^\infty}$, get the triadic rationals in $[0, 1]$.

More generally, let $(k_1, k_2, \ldots)$ be a sequence of integers ≥ 2. Set $m_0 = 1$, $m_n = \prod_{i=1}^{n} k_i$.

Set $A_n = \mathbb{M}_{m_n}$, $\phi_{n,n+1}(a) = diag(a, \ldots, a)$ (k_{n+1} times).

Let $A = \lim_{\to}(A_n, \phi_{m,n})$. A is a general UHF algebra.

Invariant is described by the formal infinite product $\prod_{i=1}^{\infty} k_i$. This is a *generalized integer* $q = 2^{r_2} 3^{r_3} 5^{r_5} \cdots$, where infinitely many nonzero exponents and infinite exponents are allowed.

The range of the trace on projections is

$$\left\{ \frac{a}{b} : 0 \leq a \leq b, \ b \text{ ``divides'' } q \right\}$$

Theorem. (Glimm) Two UHF algebras with the same invariant are isomorphic. So the UHF algebras are (truly) classified by the generalized integers.

We may write $\mathbb{M}_q$ for the UHF algebra of type q.

The $\mathbb{M}_q$ are separable, simple, unital C*-algebras. There are uncountably many distinct ones.

AF Algebras

Definition. An *AF algebra* is an inductive limit of a system of finite-dimensional C*-algebras.

Examples: $\mathbb{K}$

UHF algebras

$C_0(X)$, X zero-dimensional (e.g. the Cantor set)

AF algebras are "zero-dimensional" C*-algebras, and are a C*-analog of afd von Neumann algebras.

Except for bookkeeping complications, general AF algebras behave like UHF algebras. But invariant is clumsy to phrase in terms of traces.

Problem: Not enough traces in general to distinguish equivalence classes of projections.

Alternate approach: dimension group (K_0-group) of A

elements: formal differences of equivalence classes of projections in matrix algebras over A

positive cone: equivalence classes of projections in matrix algebras over A

scale: equivalence classes of projections in A, or class of 1_A

$K_0(A)$ is thus a *scaled ordered group* (scaled partially ordered abelian group).

Examples: (1) $K_0(\mathbb{M}_n) = (\mathbb{Z}, \mathbb{Z}_+, \{0, 1, \ldots, n\})$, or $(\mathbb{Z}, \mathbb{Z}_+, n)$

(2) $K_0(\mathbb{M}_q) = (\mathbb{Z}_{(q)}, \mathbb{Z}_{(q)+}, 1)$, where

$$\mathbb{Z}_{(q)} = \left\{ \frac{a}{b} \in \mathbb{Q} : b \text{ ``divides'' } q \right\}$$

(These are precisely the subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$.)

Theorem. (Elliott) The dimension group is a complete isomorphism invariant for AF algebras.

Theorem. (Effros-Handelman-Shen) An ordered group G is a dimension group if and only if it has the following properties:

G is countable

G is *unperforated*, i.e. $nx \geq 0$ for $n > 0 \Rightarrow x \geq 0$

G has the *Riesz interpolation property*, i.e. $x_1, x_2 \leq y_1, y_2 \Rightarrow \exists z$ with $x_1, x_2 \leq z \leq y_1, y_2$.

This gives a "classification" rather than a true classification, since the set of dimension groups is enormously complicated, even the simple dimension groups.

For example, even the countable subgroups of $\mathbb{R}$ are "unclassifiable" as ordered groups.

These theorems have proved to be very powerful as existence and uniqueness theorems.

Outline of Proof of Elliott's Theorem

A finite-dimensional C*-algebra is a direct sum of matrix algebras

$$\mathbb{M}_{k_1} \oplus \mathbb{M}_{k_2} \oplus \cdots \oplus \mathbb{M}_{k_r}$$

so its dimension group is $(\mathbb{Z}^r, \mathbb{Z}^r_+, (k_1, \ldots, k_r))$.

Theorem. (Existence) If A and B are finite-dimensional C*-algebras, and σ is a homomorphism (as scaled ordered groups) from $K_0(A)$ to $K_0(B)$, then there is a *-homomorphism $\phi : A \to B$ implementing σ ($\phi_* = \sigma$).

Theorem. (Uniqueness) If A and B are finite-dim. C*-algebras and $\phi, \psi : A \to B$ are *-homomorphisms with $\phi_* = \psi_* : K_0(A) \to K_0(B)$, then ϕ and ψ are unitarily equivalent (there is a unitary $u \in B$ such that $\psi(a) = u^*\phi(a)u\ \forall a \in A$.)

The proofs of these theorems are simple and elementary linear algebra arguments.

General scheme: If A, B are C*-algebras and σ : $(K_0(A), K_0(A)_+, \Sigma_A) \to (K_0(B), K_0(B)_+, \Sigma_B)$ is an isomorphism, first show:

Intertwining Theorem. There are homomorphisms $\alpha_k : K_0(A_{n_k}) \to K_0(B_{m_k})$ and $\beta_k : K_0(B_{m_k}) \to K_0(A_{n_{k+1}})$ making the following diagram commute:

$$K_0(A_1) \to \quad \cdots \quad \to K_0(A_{n_2}) \to \quad \cdots \quad \to K_0(A)$$

$$K_0(B_1) \to \cdots \to K_0(B_{m_1}) \to \cdots \quad \cdots \quad \to K_0(B)$$

Now use the existence and uniqueness theorems to lift this commutative diagram to get:

$$A_1 \to \quad \cdots \quad \to A_{n_2} \to \quad \cdots \quad \to A$$

$$B_1 \to \cdots \to B_{m_1} \to \cdots \quad \cdots \quad \to B$$

inducing an isomorphism $\phi : A \to B$.

E. Effros ($\approx$ 1979) proposed making a similar study of circle algebras (inductive limits of direct sums of matrix algebras over $C(\mathbb{T})$.)

No progress for about 10 years.

In 1989, I discovered pathological automorphisms of order 2 of the CAR algebra.

Crossed Products by $\mathbb{Z}_2$

If α is an automorphism of A of order 2, embed A into a larger C*-algebra $C^*(A, u)$, where $u = u^* = u^{-1}$ and $uau = \alpha(a)$.

$$A \times_\alpha \mathbb{Z}_2 = \{a + bu : a, b \in A\} \cong \left\{ \begin{bmatrix} a & \alpha(b) \\ b & \alpha(a) \end{bmatrix} : a, b \in A \right\}$$

A is a C*-subalgebra of $A \times_\alpha \mathbb{Z}_2$.

There is a dual automorphism $\widehat{\alpha}$ of $A \times_\alpha \mathbb{Z}_2$, of order 2: $\widehat{\alpha}(a) = a$, $\widehat{\alpha}(u) = -u$.

$(A \times_\alpha \mathbb{Z}_2) \times_{\widehat{\alpha}} \mathbb{Z}_2 \cong M_2(A)$ (Takai Duality).

The fixed-point algebra $A^\alpha = \{a \in A : \alpha(a) = a\}$ is stably isomorphic to $A \times_\alpha \mathbb{Z}_2$ (under mild hypotheses.)

Question: If α is an automorphism of order 2 of the CAR algebra A, are A^α and $A \times_\alpha \mathbb{Z}_2$ AF algebras?

I discovered a simple circle algebra B and an automorphism β of order 2 such that B is not AF because $K_1(B) \neq 0$ (the unitary group of B is not connected), but $A = B \times_\alpha \mathbb{Z}_2$ is isomorphic to the CAR algebra.

Thus $A \times_{\widehat{\beta}} \mathbb{Z}_2$ is not AF.

Moral: (1) Crossed products by $\mathbb{Z}_2$ can be pathological.

(2) AF algebras are not so special and different from other non-AF algebras such as circle algebras.

General Invariants

The scaled ordered K_0-group is not a complete invariant for all simple C*-algebras.

Shortcomings:

1. The unitary group of a simple C*-algebra is not connected in general.

2. There might not be enough projections in the algebra:

There may not be "small" projections

Projections need not distinguish traces

3. The algebra may be "too large" (nonseparable) or the internal structure of the algebra may be pathological.

Example. Embed $M_{k_n}(C(\mathbb{T}))$ into $M_{k_{n+1}}(C(\mathbb{T}))$, where $k_{n+1} = (m_n + 1)k_n$, by

$$[\phi_{n,n+1}(f)](z) = diag(f(z), \ldots, f(z), f(z_n))$$

where z_n runs over a dense set in $\mathbb{T}$. The inductive limit A is simple. If $k_n \to \infty$ rapidly (e.g. $k_n = n^2$), A has an infinite-dimensional trace space. $K_0(A)$ and $K_1(A)$ are dense subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$.

Example. (Jiang-Su) If $p, q \in \mathbb{N}$, write $\mathbb{M}_{pq}$ as a tensor product $\mathbb{M}_p \otimes \mathbb{M}_q$, and let

$$D_{p,q} = \{f : [0,1] \to \mathbb{M}_{pq} | f(0) \in \mathbb{M}_p \otimes 1,\ \ f(1) \in 1 \otimes \mathbb{M}_q\}$$

If p, q relatively prime, $D_{p,q}$ is projectionless.

For suitable relatively prime p_n, q_n, there are embeddings $\phi_{n,n+1}$ of D_{p_n,q_n} into $D_{p_{n+1},q_{n+1}}$ so that the inductive limit D is simple with unique trace. The construction can be varied to give nonunique traces.

To overcome (3), restrict attention to separable C*-algebras which are *nuclear* (approximately finite-dimensior in an order-theoretic sense.)

All type I C*-algebras are nuclear

The class of nuclear C*-algebras is closed under most standard operations (inductive limits, tensor products, extensions, crossed products by amenable groups)

"Most" C*-algebras "arising naturally" are nuclear.

Nuclear C*-algebras are also precisely the *amenable* C*-algebras, and are characterized by other natural conditions (e.g. nice behavior under tensor products.)

Nuclear C*-algebras are the most natural C*-analog of the afd (injective) von Neumann algebras.

A is nuclear $\Leftrightarrow$ $\pi(A)''$ is injective for every representation π of A.

To overcome (1), add K_1 to the invariant. [Roughly, $K_1(A)$ = (unitary group)/(connected component of 1).]

There is a natural ordering on

$$K_*(A) = K_0(A) \oplus K_1(A)$$

yielding a scaled ordered K-group for A.

To overcome (2), add the trace space $T(A)$ to the invariant.

Definition. A *state* on a scaled ordered group (G, G_+, u) is an order-preserving homomorphism $f : G \to \mathbb{R}$ with $f(u) = 1$.

$S(G)$ = set of states of G

There is a map $\chi : T(A) \to S(K_0(A))$.

χ is surjective if A is nuclear. (B.-Rørdam-Haagerup)

"Definition." A C*-algebra has *real rank zero* if it has "many" projections.

Examples: (1) AF algebras

(2) $C_0(X)$ has real rank zero $\Leftrightarrow X$ is zero-dimensional.

Real rank zero is the noncommutative analog of zero-dimensionality.

$\chi : T(A) \to S(K_0(A))$ is a bijection if A has real rank zero.

Elliott Invariant: $(K_*(A), T(A), \chi)$ (must be slightly modified in the nonunital case.)

$K_*(A) = K_0(A) \oplus K_1(A)$ is a scaled (pre)ordered group.

Elliott's Conjecture: The Elliott invariant is a complete isomorphism invariant for infinite-dimensional separable simple nuclear C*-algebras.

The infinite-dimensional condition is necessary: the Jiang-Su example has the same Elliott invariant as $\mathbb{C}$.

Finiteness

Definition. A projection is *finite* if it is not equivalent to a proper subprojection.

A is *finite* if 1_A is finite.

A is *stably finite* if $M_n(A)$ is finite for all n.

A is *purely infinite* if every nonzero positive element dominates an infinite projection.

$T(A)$ nonempty $\Rightarrow$ A stably finite. Converse true in nuclear case.

Question: Is every simple C*-algebra either stably finite or purely infinite?

Classification theorems apply in these two cases.

Stably Finite Case

Program: study and classify (simple) inductive limits of well-understood building blocks:

AF algebras: finite-dimensional building blocks

AH algebras: homogeneous building blocks

ASH algebras: subhomogeneous building blocks

Base spaces are restricted:

- intervals
- circles
- graphs
- two-dimensional CW complexes
- three-dimensional CW complexes
- bounded dimension (slow dimension growth)

Scheme of Proof

Modeled after AF case. First prove:

Intertwining Theorem. If $A = \lim A_n$ and $B = \lim B_n$ with A_n, B_n building blocks, there is a compatible (approximate) intertwining

$$K_*(A_1) \to \quad \cdots \quad \to K_*(A_{n_2}) \to \quad \cdots \quad \to K_*(A)$$

$$K_*(B_1) \to \cdots \to K_*(B_{m_1}) \to \cdots \quad \cdots \quad \to K_*(B)$$

$$T(A_1) \leftarrow \quad \cdots \quad \leftarrow T(A_{n_2}) \leftarrow \quad \cdots \quad \leftarrow T(A)$$

$$T(B_1) \leftarrow \cdots \leftarrow T(B_{m_1}) \leftarrow \cdots \quad \cdots \quad \leftarrow T(B)$$

(Generally routine)

Then prove existence and uniqueness theorems to allow lifting to an (approximate) intertwining

$$A_1 \to \quad \cdots \quad \to A_{n_2} \to \quad \cdots \quad \to A$$

$$B_1 \to \cdots \to B_{m_1} \to \cdots \qquad \cdots \quad \to B$$

This is the major work.

New feature: Approximate homomorphisms on both algebra and invariant level

Partially defined (on invariant)

Asymptotically commuting diagrams

Asymptotically multiplicative homomorphisms (on algebras)

Current Status

Simple case: (Elliott, Gong, Li, Su, Thomsen, ...)

Up to perforation, all possible values of the Elliott invariant in unital stably finite case can be realized within the classified class.

Real rank zero ASH algebras are "nearly finished"

Non-real rank zero case has farther to go.

Villadsen: Perforation can occur in non-rank-zero case. Unknown in real rank zero case.

All possible values of the invariant can occur among ASH algebras, but classification barely begun.

Some work also done on stably projectionless case.

Nonsimple Case: (Dadarlat, Eilers, Gong, Loring, ...)

Invariant must be expanded

Add primitive ideal space

Expand K_* to $\mathbf{K}_*$ (K-theory with coefficients)

Big Problem: Is every stably finite simple unital nuclear C*-algebra an ASH algebra?

Hard to get a handle on this.

Partial results: some unexpected examples turn out to be ASH (irrational rotation algebras, etc.)

Theorem. (Q. Lin-Phillips) If α is a minimal diffeomorphism of a closed manifold X, then $C(X) \times_\alpha \mathbb{Z}$ is ASH and in the classified class.

H. Lin has also expanded the classified class to include many simple C*-algebras not obviously AF ("Tracially AF", etc.) Related work by Dadarlat.

E. Kirchberg and I have been studying generalized inductive limits of finite-dimensional C*-algebras. One of our results:

Theorem. Every separable simple quasidiagonal nuclear C*-algebra can be written as an inductive limit of residually finite-dimensional nuclear C*-algebras.

Bivariant K-Theory

Kasparov developed a bivariant K-theory $KK^*(\cdot,\cdot)$

$$KK^*(\mathbb{C}, B) \cong K_*(B)$$

$KK^*(A, \mathbb{C})$ is "K-homology" of A (Brown-Douglas-Fillmore $Ext(A)$)

Elements of $KK(A, B)$ are (homotopy classes of formal differences of) "quasihomomorphisms" from A to B ("Fredholm modules")

Connes-Higson $E(A, B)$ is the group of homotopy classes of "asymptotic homomorphisms" from SA to SB

$E(A, B) = KK(A, B)$ if A nuclear.

Key feature: product ("composition")

$$KK(A, B) \times KK(B, C) \to KK(A, C)$$

Allows making KK (or E) into a category.

KK-Equivalence: isomorphism in this category.

Universal Coefficient Theorem

There is a natural homomorphism

$$\gamma : KK^*(A, B) \to \text{Hom}(K_*(A), K_*(B))$$

and a map from $ker\ \gamma$ to $\text{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$.

A satisfies the Rosenberg-Schochet *Universal Coefficient Theorem* if the following sequence is exact for all B:

$$0 \to \text{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to KK^*(A, B)$$

$$\to \text{Hom}(K_*(A), K_*(B)) \to 0$$

The class of nuclear C*-algebras satisfying the UCT is called the *bootstrap class*. It includes Type I C*-algebras andis closed under extensions, inductive limits, tensor products, and crossed products by $\mathbb{Z}$.

The bootstrap class contains all known separable nuclear C*-algebras.

Question. Is every separable (simple) nuclear C*-algebra in the bootstrap class?

Purely Infinite Case

Purely infinite $\Rightarrow T(A)$ is empty and $K_0(A)_+ = K_0(A)$

Purely infinite $\Rightarrow$ real rank zero

So Elliott invariant just becomes $(K_0(A), K_1(A))$, a pair of countable abelian groups with a distinguished element of K_0 (scale)

Every possible such pair occurs (Rørdam)

Examples: Cuntz algebras O_n, Cuntz-Krieger algebras O_A, many graph algebras (Kumjian-Pask *et al.*)

E. Kirchberg (and N. C. Phillips) proved the following remarkable theorem:

Theorem. If A and B are separable simple purely infinite C*-algebras which are KK-equivalent, then A and B are stably isomorphic ($A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.)

Corollary. The Elliott invariant is a complete isomorphism invariant for purely infinite simple unital C*-algebras in the bootstrap class for the Universal Coefficient Theorem of K-theory.

Future Projects

1. Solve the UCT problem.

2. Decide whether every simple C*-algebra is either stably finite or purely infinite.

3. Extend the classification to include Villadsen's examples, and stably projectionless simple C*-algebras.

4. Give intrinsic conditions for a separable nuclear C*-algebra to be in the classifiable class.

5. Extend the purely infinite classification to the nonsimple case.

6. Consider classes of nonnuclear C*-algebras, e.g. exact C*-algebras.

Exact C*-algebras are a very natural class. They coincide with the subnuclear C*-algebras. "Most" group C*-algebras are exact. Many reduced free product C*-algebras are exact.