NUCLEARITY, QUASIDIAGONALITY, AND GENERALIZED INDUCTIVE LIMITS

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Inductive Limits

Definition. An *inductive system* of C*-algebras (over $\mathbb{N}$) is a sequence (A_n) of C*-algebras and connecting *-homomorphisms $\phi_{n,n+1} : A_n \to A_{n+1}$ (not necessarily injective.)

If $(A_n, \phi_{n,n+1})$ is an inductive system, put a seminorm on the algebraic direct limit by

$$\|\phi_m(a)\| = \inf_{n \geq m} \|\phi_{m,n}(a)\|$$

The completion is called the *inductive limit*, denoted $\lim_{\to}(A_n, \phi_{m,n})$.

If each $\phi_{n,n+1}$ is injective, $\lim_{\to}(A_n, \phi_{m,n})$ is the completion of the "union" of the A_n.

Can embed $\lim_{\to}(A_n, \phi_{m,n})$ into $(\prod A_n)/(\oplus A_n)$:

$$\phi_m(a) = \pi(\ldots, \phi_{m,n}(a), \ldots)$$

Closure of $\cup \phi_n(A_n)$ is $\lim_{\to}(A_n, \phi_{m,n})$.

Generalized Inductive Systems

Definition. A *generalized inductive system of C*-algebras* is a sequence (A_n) of C*-algebras, with coherent maps $\phi_{m,n} : A_m \to A_n$ for $m < n$, such that for all k and all $x, y \in A_k$, $\lambda \in \mathbb{C}$, and all $\epsilon > 0$, there is an M such that, for all $M \leq m < n$,

(1) $\|\phi_{m,n}(\phi_{k,m}(x) + \phi_{k,m}(y)) - (\phi_{k,n}(x) + \phi_{k,n}(y))\| < \epsilon$

(2) $\|\phi_{m,n}(\lambda\phi_{k,m}(x)) - \lambda\phi_{k,n}(x)\| < \epsilon$

(3) $\|\phi_{m,n}(\phi_{k,m}(x)^*) - \phi_{k,n}(x)^*\| < \epsilon$

(4) $\|\phi_{m,n}(\phi_{k,m}(x)\phi_{k,m}(y)) - \phi_{k,n}(x)\phi_{k,n}(y)\| < \epsilon$

(5) $\sup_r \|\phi_{k,r}(x)\| < \infty$

A system satisfying (1) [resp. (4)] is called *asymptotically additive* [resp. *asymptotically multiplicative*]. A generalized inductive system in which all $\phi_{m,n}$ are linear is called a *linear generalized inductive system*; if all the $\phi_{m,n}$ also preserve adjoints, the system is called **-linear*. A system is *contractive* if all the connecting maps are contractions.

Generalized Inductive Limits

If $(A_n, \phi_{m,n})$ is a generalized inductive system, define $\phi_m : A_m \to (\prod A_n)/(\oplus A_n)$ by

$$\phi_m(a) = \pi(\ldots, \phi_{m,n}(a), \ldots)$$

Closure of $\cup \phi_n(A_n)$ is a C*-algebra $\lim_{\to}(A_n, \phi_{m,n})$.

Since $\lim_{\to}(A_n, \phi_{m,n})$ is a C*-subalgebra of $(\prod A_n)/(\oplus A_n)$,

If each A_n is commutative, $\lim_{\to}(A_n, \phi_{m,n})$ is commutative.

If each A_n is [stably] finite, $\lim_{\to}(A_n, \phi_{m,n})$ is [stably] finite.

If each A_n has a tracial state, and each $\phi_{m,n}$ is unital, then $\lim_{\to}(A_n, \phi_{m,n})$ has a tracial state.

MF Algebras

A separable C*-algebra is an *MF algebra* if it is isomorphic to a generalized inductive limit $\lim_{\rightarrow}(A_n, \phi_{m,n})$ with each A_n finite-dimensional.

Theorem. Let A be a separable C*-algebra. Then the following are equivalent:

(i) A is an MF algebra

(ii) A is isomorphic to $\lim_{\rightarrow}(A_n, \phi_{m,n})$ for a *-linear generalized inductive system of finite-dimensional C*-algebras

(iii) A can be embedded as a C*-subalgebra of the corona algebra $(\prod \mathbb{M}_{k_n})/(\oplus \mathbb{M}_{k_n})$ for some sequence $\langle k_n \rangle$

(iv) There is a continuous field of C*-algebras $\langle B(t) \rangle$ over $\mathbb{N} \cup \{\infty\}$ with $B(\infty) = A$ and $B(n)$ finite-dimensional for $n < \infty$.

(v) There is a continuous field of C*-algebras $\langle B(t) \rangle$ over $\mathbb{N} \cup \{\infty\}$ with $B(\infty) = A$ and $B(n) = \mathbb{M}_{k_n}$ for $n < \infty$.

Properties of MF Algebras

Every C*-subalgebra of an MF algebra is MF.

Definition. A C*-algebra is *residually finite-dimensional* if it has a separating family of finite-dimensional representations.

Every residually finite-dimensional C*-algebra is an MF algebra.

In particular, every subhomogeneous C*-algebra is an MF algebra.

Not every MF algebra is residually finite-dimensional - many AF algebras are not residually finite-dimensional.

Every separable C*-algebra is a quotient of an MF algebra.

Any generalized inductive limit of MF algebras is an MF algebra.

Every MF algebra is stably finite.

It is hard to give an example of a stably finite separable C*-algebra which is not MF. $C^*(G)$ for a group G which has Property T but is not maximally almost periodic is an example.

Quasidiagonality

If $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$, then $\mathcal{S}$ is a *quasidiagonal set of operators* if the operators in $\mathcal{S}$ are simultaneously block-diagonalizable up to compacts.

Proposition. Let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$. Then the following are equivalent:

(i) $\mathcal{S}$ is a quasidiagonal set of operators.

(ii) There is an increasing net (P_i) of finite-rank projections on $\mathcal{H}$, with $\bigvee_i P_i = I$ (i.e. $P_i \to I$ strongly), such that $\lim_i \|[P_i, S]\| = 0$ for all $S \in \mathcal{S}$.

(iii) For every finite-rank projection $Q \in \mathcal{L}(\mathcal{H})$, $S_i, \ldots, S_n \in \mathcal{S}$, and $\epsilon > 0$, there is a finite-rank projection $P \in \mathcal{L}(\mathcal{H})$ with $Q \leq P$ and $\|[P, S_i]\| < \epsilon$ for $1 \leq i \leq n$.

Condition (ii) or (iii) is frequently taken as the definition of a quasidiagonal set of operators.

Corollary. Let A be a concrete C*-algebra of operators on $\mathcal{H}$, containing $\mathcal{K}(\mathcal{H})$. Then A is a quasidiagonal C*-algebra of operators if and only if there is an approximate unit for $\mathcal{K}(\mathcal{H})$, consisting of projections, which is quasicentral for A.

Quasidiagonal C*-Algebras

Definition. A C*-algebra is *quasidiagonal* if it has a faithful representation as a quasidiagonal algebra of operators.

If A is quasidiagonal, then any faithful representation of A not intersecting $\mathbb{K}$ is quasidiagonal (Voiculescu's Weyl-von Neumann Theorem)

Theorem. A C*-algebra A is quasidiagonal if and only if, for every $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a representation π of A on a Hilbert space $\mathcal{H}$ and a finite-rank projection $P \in \mathcal{L}(\mathcal{H})$ with $\|P\pi(x_j)P\| > \|x_j\| - \epsilon$ and $\|[P, \pi(x_j)]\| < \epsilon$ for $1 \leq j \leq n$.

Corollary. A separable C*-algebra A is quasidiagonal if and only if there is a completely positive contraction $\phi : A \to \prod \mathbb{M}_{k_n}$, for some sequence $\langle k_n \rangle$, with $\pi \circ \phi : A \to (\prod \mathbb{M}_{k_n})/(\oplus \mathbb{M}_{k_n})$ a *-homomorphism.

Quasidiagonal Extensions

A C*-algebra A has a quasidiagonal extension by $\mathbb{K}$ if there is a quasidiagonal C*-algebra of operators B, containing $\mathbb{K}$, with $B/\mathbb{K} \cong A$.

A quasidiagonal C*-algebra has a split quasidiagonal extension by $\mathbb{K}$.

Corollary. A C*-algebra is quasidiagonal if and only if it has a semisplit quasidiagonal extension by $\mathbb{K}$.

Semisplit means completely positive cross section $A \to B$.

Proposition. A C*-algebra has a quasidiagonal extension by $\mathbb{K}$ if and only if it embeds in $(\prod \mathbb{M}_{k_n})/\oplus\mathbb{M}_{k_n})$ for some sequence $\langle k_n \rangle$.

Corollary. A C*-algebra is an MF algebra if and only if it has a quasidiagonal extension by $\mathbb{K}$.

Nuclear C*-Algebras

There are many equivalent definitions. We will use:

Definition. A C*-algebra A is *nuclear* if the identity map on A can be approximated in the point-norm topology by completely positive contractions through finite-dimensional C*-algebras [matrix algebras], i.e. given $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra [matrix algebra] B and completely positive contractions $\alpha : A \to B$ and $\beta : B \to A$ such that $\|x_i - \beta \circ \alpha(x_i)\| < \epsilon$ for $1 \leq i \leq n$.

$$A \quad \xrightarrow{id} \quad A$$

$$B$$

NF Algebras

Definition. A separable C*-algebra is an *NF algebra* if it can be written as a generalized inductive limit $\lim_{\to}(A_n, \phi_{m,n})$, with each A_n finite-dimensional and $\phi_{m,n}$ a completely positive contraction.

Theorem. Let A be a separable C*-algebra. The following are equivalent:

(i) A is an NF algebra

(ii) A is a nuclear MF algebra

(iii) A is nuclear and can be embedded as a C*-subalgebra of $(\prod \mathbb{M}_{k_n})/(\oplus \mathbb{M}_{k_n})$ for some sequence $\langle k_n \rangle$

(iv) A is nuclear and quasidiagonal

(v) The identity map on A can be approximated in the point-norm topology by completely positive approximately multiplicative contractions through finite-dimensional C*-algebras, i.e.

Given $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra B and completely positive contractions $\alpha : A \to B$ and $\beta : B \to A$ such that $\|x_i - \beta \circ \alpha(x_i)\| < \epsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \epsilon$ for all i, j.

$$\begin{array}{ccc} A & \xrightarrow{id} & A \\ & & \\ & B & \end{array}$$

Thus the NF algebras form a very natural class of nuclear C*-algebras, the ones in which not only the complete order structure but also the multiplication can be approximately modeled in finite-dimensional C*-algebras.

A nuclear C*-subalgebra of an NF algebra is NF.

An inductive limit of NF algebras is NF.

If A is any nuclear C*-algebra, then CA and SA are quasidiagonal (Voiculescu) and hence NF.

There is no known stably finite nuclear C*-algebra which is not NF.

Cannot assume β is approximately multiplicative unless A is an AF algebra.

If B is a finite-dimensional C*-subalgebra of A, there is a conditional expectation from A to B.

More generally, if $\phi : B \to A$ is a complete order embedding, there is an idempotent completely positive contraction θ from A onto $\phi(B)$.

Nonstandard definition of AF algebras:

Definition. A C*-algebra is an *AF algebra* if, for any $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra B and completely positive contractions $\alpha : A \to B$ and $\beta : B \to A$ such that $\|x_i - \beta \circ \alpha(x_i)\| < \epsilon$ and β is an injective *-homomorphism.

It follows that the α's are automatically asymptotically multiplicative in the sense of NF algebras.

Proposition. (Choi-Effros) Let A be a C*-algebra and $\theta : A \to A$ an idempotent completely positive contraction. Then the product $x \cdot y = \theta(xy)$ makes $\theta(A)$ into a C*-algebra B, and $\theta|_{C^*(\theta(A))}$ is a *-homomorphism.

Strong NF Algebras

Definition. A separable C*-algebra is a *strong NF algebra* if it can be written as a generalized inductive limit $\lim_{\rightarrow}(A_n, \phi_{m,n})$, with each A_n finite-dimensional and $\phi_{m,n}$ a complete order embeddings.

There is then automatically an idempotent completely positive contraction $\gamma_{n+1,n} : A_{n+1} \rightarrow \phi_{n,n+1}(A_n)$ for each n, yielding an idempotent c.p. contraction $A \rightarrow \phi_n(A_n)$. These maps are asymptotically multiplicative.

AF $\Rightarrow$ Strong NF $\Rightarrow$ NF

The implications cannot be reversed.

Every commutative C*-algebra is a strong NF algebra.

A unital c.p. contraction from $\mathbb{C}^n$ to $\mathbb{C}$ is a convex combination of homomorphisms.

A (unital) complete order embedding from $\mathbb{C}^n$ to $\mathbb{C}^m$ gives a surjective affine map from Δ_{m-1} (or any subcomplex) to Δ_{n-1}.

A (unital) complete order embedding from $\mathbb{C}^n$ to $C(X)$ corresponds to a "triangulation" of X, i.e. a choice of a partition of unity.

$$(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{k=1}^{n} \lambda_k f_k$$

A strong NF system for $C(X)$ is equivalent to writing $X = \lim_{\leftarrow} X_n$, where the X_n are simplicial complexes with increasingly fine triangulations, and the connecting maps are piecewise linear.

Study of strong NF algebras is "noncommutative PL-topology."

Theorem. Let A be a separable C*-algebra. The following are equivalent:

(i) A is a strong NF algebra

(ii) There is an increasing sequence (S_n) of finite-dimensional *-subspaces of A, each completely order isomorphic to a (finite-dimensional) C*-algebra, with dense union

(iii) Given $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra B, a complete order embedding ϕ of B into A, and elements $b_1, \ldots, b_n \in B$ with $\|x_i - \phi(b_i)\| < \epsilon$ for $1 \leq i \leq n$

(iv) The identity map on A can be approximated in the point-norm topology by idempotent completely positive finite-rank contractions from A to A, i.e. given $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is an idempotent completely positive finite-rank contraction $\theta : A \to A$ with $\|x_i - \theta(x_i)\| < \epsilon$ for $1 \leq i \leq n$.

(v) The identity map on A can be approximated in the point-norm topology by completely positive approximately multiplicative retractive contractions through finite-dimensional C*-algebras, i.e. given $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra B and completely positive contractions $\alpha : A \to B$ and $\beta : B \to A$ with $\alpha \circ \beta = id_B$ (β is then automatically a complete order embedding), such that $\|x_i - \beta \circ \alpha(x_i)\| < \epsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \epsilon$ for all i, j.

(vi) Same as (v) with the "approximately multiplicative" condition on α deleted.

(vii) Given $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra B and completely positive contractions $\alpha : A \to B$ and $\beta : B \to A$ with β a complete order embedding, such that $\|x_i - \beta \circ \alpha(x_i)\| < \epsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \epsilon$ for all i, j.

(viii) Same as (vii) with the "approximately multiplicative" condition on α deleted.

(ix) A is nuclear and inner quasidiagonal (has a separating family of quasidiagonal irreducible representations.)

Thus not every NF algebra is strong NF:

Examples. $C^*(S \oplus S^*)$, S the unilateral shift

SO_2

Every primitive NF algebra is strong NF. In particular, every simple NF algebra is strong NF.

If A is NF and π is a faithful representation of infinite multiplicity, then $\pi(A) + \mathbb{K}$ is strong NF. So every NF algebra has a split extension by $\mathbb{K}$ which is strong NF. In particular, every NF algebra embeds in a strong NF algebra.

An inductive limit of strong NF algebras with *injective* connecting maps is strong NF.

A generalized inductive limit of strong NF algebras with complete order embeddings as connecting maps is strong NF.

A residually finite-dimensional nuclear C*-algebra is strong NF. Conversely:

Theorem. Every strong NF algebra is an (*ordinary*) inductive limit of residually finite-dimensional C*-algebras.

Roughly: if $A = \lim_{\rightarrow}(A_n, \phi_{m,n})$ is a strong NF algebra, then $C^*(\phi_m(A_m))$ is residually finite-dimensional. But not obviously nuclear.

Instead, for each m inductively for $n \geq m$ define $C_{m,n} \subseteq A_n$ by $C_{m,m} = A_m$, $C_{m,n+1} = C^*(\phi_{n,n+1}(C_{m,n}))$, and $C_m = [\cup \phi_n(C_{m,n})]^- \subseteq A$.

$C_m = \lim_{\rightarrow}(C_{m,n}, \phi_{m,n})$, so C_m is a strong NF algebra and hence nuclear.

The map $\gamma_n : A \to A_n$ is a *-homomorphism from C_m onto $C_{m,n}$ for each n, and these separate points, so C_m is residually finite-dimensional. $A = [\cup C_m]^-$.

Open Questions

1. Is every stably finite nuclear C*-algebra an NF algebra?

2. Can every (strong) NF algebra be embedded in an AF algebra?

(Partial results by Dadarlat)

3. Is there an effective way to compute the K-theory and trace space of a (strong) NF algebra?

"Noncommutative Čech cohomology"

4. Universal Coefficient Theorem?

Proposition. If every residually finite nuclear C*-algebra satisfies the UCT (is in the bootstrap class), then every nuclear C*-algebra is in the bootstrap class.

If A is nuclear, then $S^2A + \mathbb{K}$ is strong NF and is an inductive limit of residually finite-dimensional C*-algebras C_n. A satisfies the UCT if all the C_n do.