Symmetry and Complex Structure in C*-Algebras

Bruce Blackadar

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C*-Algebras

Recall the definition of a C*-algebra:

Definition:

A C*-algebra is a complex Banach algebra A, with an involution * satisfying

$$(x+y)^* = x^* + y^*\,,\ (\lambda x)^* = \bar{\lambda} x^*\,,\ (xy)^* = y^* x^*\,,\ (x^*)^* = x$$

for all $x, y \in A, \lambda \in \mathbb{C}$, and satisfying the C*-axiom

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The key point for our discussion is that the complex scalar multiplication must be specified as part of the C*-algebra structure.

-Homomorphisms between C-algebras must respect the scalar multiplication.

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A with this new scalar multiplication is a *different* C*-algebra, called the *conjugate C*-algebra* of A, denoted A^c.

The identity map from A to A^c is a *-isomorphism of *real* C*-algebras. But A^c need not be isomorphic to A as complex C*-algebras.

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If A has central projections, there are more than two possibilities: orientations can be chosen separately on direct summands.

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Definition

A C*-algebra A is *symmetric* if $A \cong A^c$ (as complex C*-algebras).

More familiar construction:

Keep the addition, involution, scalar multiplication, and norm on A, and reverse the multiplication:

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A^c is isomorphic to A^{op} as a (complex) C*-algebra via $x \mapsto x^*$, hence is anti-isomorphic to A as a (complex) C*-algebra. A and A^{op} are *-isomorphic as real C*-algebras via $x \mapsto x^*$. But A and A^{op} need not be isomorphic as (complex) C*-algebras (Connes, Phillips, B.)

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It is obvious that $A \mapsto A^c$ (or $A \mapsto A^{op}$) is functorial, i.e. a (complex-linear) *-homomorphism from A to B gives a (complex-linear) *-homomorphism from A^c to B^c.

Stable Homogeneous C*-Algebras

Stable homogeneous C*-algebras were classified by Dixmier and Douady in the 1960s.

Ingredients: a base space X, assumed compact and metrizable for simplicity

An element $\sigma \in H^3(X)$, called the *Dixmier-Douady invariant*.

Theorem.

(i) To each such pair, a separable stable homogeneous C*-algebra $C^*(X, \sigma)$ is associated. Every separable stable homogeneous C*-algebra A with $\mathrm{Prim}(A) \cong X$ is of this form.

(ii) $C^*(X, \sigma) \cong C^*(Y, \tau)$ if and only if there is a homeomorphism $\phi : X \to Y$ with $\phi^*(\tau) = \sigma$.

Simplest case: X is a closed connected orientable 3-manifold. Then $H^3(X) \cong \mathbb{Z}$. The stable continuous trace C*-algebras over X are thus of the form $C^*(X, n)$ for $n \in \mathbb{Z}$.

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$C^*(X, n)$ is symmetric if and only if there is a homeomorphism of X which sends n to $-n$, i.e. an orientation-reversing homeomorphism if $n \neq 0$.

Lens Spaces

Regard S^3 as the unit sphere in $\mathbb{C}^2$. Let $p \in \mathbb{N}$, $p \geq 2$, and $1 \leq q \leq p-1$ relatively prime to p. Define an action of $\mathbb{Z}_p$ on S^3 by letting the generator act by

$$(z, w) \mapsto (e^{2\pi i/p} z, e^{2\pi iq/p} w) \ .$$

This is a free action. Denote the quotient, which is a closed connected oriented 3-manifold, by $L(p, q)$. The usual orientation on S^3 induces a canonical orientation on $L(p, q)$.

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Lens spaces have been classified up to homeomorphism, homotopy equivalence, oriented homeomorphism, and oriented homotopy equivalence.

$L(7,2)$ and $L(7,3)$ are homeomorphic but not oriented homeomorphic. Thus $L(7,2)$ has no orientation-reversing homeomorphism.

Thus, if $X = L(7,2)$ and $n \neq 0$ (say $n = 1$), then $A = C^*(X, n)$ is not symmetric.

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A standard iterative construction, obtained by taking automorphisms (θ_n) of A induced by a minimal sequence (ϕ_n) of automorphisms of X homotopic to the identity and successively embedding A into $M_2(A) \cong A$ by

$$x \mapsto diag(x, \theta_n(x))$$

a projectionless simple C*-algebra is obtained which is possibly not symmetric.

More sophisticated versions of the construction yield more promising examples.

K-Theory

$K_*(A)$ and $K_*(A^c)$ (or $K_*(A^{op})$) are naturally isomorphic since the projections, unitaries, and partial isometries in A^c (and A^{op}) are just the projections, unitaries, and partial isometries in A respectively.

Thus K-theory (at least the K-groups) does not distinguish between A and A^c: K-theory does not "see" the complex structure of A.

However, functional calculus depends on the scalar multiplication, so is different in A^c than in A. The spectrum of an element $x \in A$ is also different in A and A^c in general. (Functional calculus and spectrum are the same in A and A^{op}.)

As a result, if the complex scalar multiplication is conjugated, the connecting maps in the six-term exact sequence of K-theory change sign. So the complex scalar multiplication is subtly encoded in K-theory in the signs of these terms.

Bivariant K-Theory

There are many variations of bivariant K-theory:

KK-theory

E-theory

Cuntz's general bivariant theories

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Cuntz's general bivariant theories

In their abstract formulations and concrete realizations (using Fredholm modules, asymptotic morphisms, etc.), the complex structure is used, so it is natural to expect that these theories may "see" the complex structure of the algebras.

Thus they may give finer structure invariants than ordinary K-theory.

Recall the definition of a (pre-)Hilbert B-module:

Definition

A *pre-Hilbert B-module* is a right B-module $\mathcal{E}$ which is compatibly a complex vector space, equipped with a *B-valued pre-inner product* $\langle \cdot\,,\cdot\rangle : \mathcal{E} \times \mathcal{E} \to B$ with the following properties for $\xi, \eta, \zeta \in \mathcal{E}$, $b \in B$, $\lambda \in \mathbb{C}$:

(i) $\langle \xi, \eta + \zeta\rangle = \langle \xi, \eta\rangle + \langle \xi, \zeta\rangle$ and $\langle \xi, \lambda\eta\rangle = \lambda\langle \xi, \eta\rangle$

(ii) $\langle \xi, \eta b\rangle = \langle \xi, \eta\rangle b$

(iii) $\langle \eta, \xi\rangle = \langle \xi, \eta\rangle^*$

(iv) $\langle \xi, \xi\rangle \geq 0$ (as an element of B).

"Compatible" means:

$$(\lambda\xi)b = \lambda(\xi b) = \xi(\lambda b)$$

for all $\lambda \in \mathbb{C}$, $\xi \in \mathcal{E}$, $b \in B$.

The compatible complex-linearity is perhaps underappreciated. If this is weakened to real-linearity, the larger class of real (pre-)Hilbert B-modules is obtained, and KK-theory using this larger class of Hilbert modules is Kasparov's real KK-theory $KK_{\mathbb{R}}$.

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If $\mathcal{E}$ is a Hilbert B-module, the compatible scalar multiplication is in a sense implicit, since if $\mathcal{E}$ is a right B-module with a complete definite inner product satisfying (ii), (iii), (iv), and the first half of (i), then there is a unique compatible scalar multiplication on $\mathcal{E}$ defined by

$$\lambda\xi = \lim \xi(\lambda h_i)$$

where (h_i) is an approximate unit for B.

So even though the scalar multiplication on $\mathcal{E}$ is completely determined by the scalar multiplication on B, it is an important part of the structure of $\mathcal{E}$. If we want to regard $\mathcal{L}(\mathcal{E})$ as a C*-algebra, we must use the scalar multiplication on $\mathcal{L}(\mathcal{E})$ induced by the scalar multiplication on $\mathcal{E}$. A representation of a C*-algebra as operators on $\mathcal{E}$ must respect this scalar multiplication.

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If $\mathcal{E}$ is a Hilbert B-module, then $\mathcal{E}$ with the same addition, module structure, and inner product, but with scalar multiplication conjugated, is naturally a Hilbert B^c-module we denote $\mathcal{E}^c$. A $T \in \mathcal{L}(\mathcal{E})$ gives an adjointable operator on $\mathcal{E}^c$, and the "identity map" gives an identification of $[\mathcal{L}(\mathcal{E})]^c$ with $\mathcal{L}(\mathcal{E}^c)$.

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We thus get a natural identification of $KK(A, B)$ with $KK(A^c, B^c)$ for any A and B, which is natural even at the level of Kasparov modules. In particular, for any A and B, $KK(A^c, B)$ is naturally isomorphic to $KK(A, B^c)$.

Main Question:

Does there exist a separable (nuclear) C*-algebra A which is not KK-equivalent to A^c?

In other words: Does KK "see" the complex structure of A?

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Alternate Formulation: $A \mapsto A^c$ gives an involution on the category of separable C*-algebras and *-homomorphisms, and also on the category **KN** whose objects are separable nuclear C*-algebras and for which the morphisms from A to B are the elements of $KK(A, B)$, with composition via the Kasparov product.

On the full subcategory **KKN** with objects in the bootstrap class $\mathcal{N}$, this involution is "trivial" (preserves isomorphism classes).

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Question:

Is this involution trivial on **KN**?

The UCT Revisited

The general validity of the Universal Coefficient Theorem for separable nuclear C*-algebras is the "elephant in the attic" in the subject of Classification. Many results include validity of the UCT as a hypothesis.

We will break down this question into several subquestions and interpret the previous problems as a possible obstruction.

The Universal Coefficient Theorem

Statement: For many pairs (A, B) of separable C*-algebras, there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

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The class $\mathcal{N}$ of all separable nuclear C*-algebras A for which this sequence is valid for all separable C*-algebras B is called the *Bootstrap Class*.

Theorem (Rosenberg-Schochet):

The class $\mathcal{N}$ is precisely the smallest class of separable nuclear C*-algebras containing $\mathbb{C}$ and closed under several standard bootstrap operations. $\mathcal{N}$ is also precisely the class of separable nuclear C*-algebras which are KK-equivalent to commutative C*-algebras.

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UCT Question:

Is $\mathcal{N}$ the class of all separable nuclear C*-algebras?

Recall where the maps γ and δ come from:

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For γ:
If $\mathbf{x} \in KK(A, B)$, then using the pairing

$$KK(\mathbb{C}, A) \times KK(A, B) \to KK(\mathbb{C}, B)$$

and the isomorphism $K_0(D) \cong KK(\mathbb{C}, D)$, right Kasparov product with $\mathbf{x}$ gives a homomorphism from $K_0(A)$ to $K_0(B)$.

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Similarly, a map from $K_1(A) \cong KK(S\mathbb{C}, A)$ to $K_1(B) \cong KK(S\mathbb{C}, B)$ is obtained.

The δ is more indirect. There is a natural identification of $KK(A, B)$ with $Ext(A, SB)^{-1}$, the group of invertible extensions of A by $SB \otimes \mathbb{K}$. Any such extension E defines a six-term cyclic exact sequence

$$\begin{array}{ccccc} K_0(SB) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ \partial \big\uparrow & & & & \big\downarrow \partial \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(SB) \end{array}$$

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The connecting maps ∂ are exactly the maps induced by γ. If $\mathbf{x} \in ker(\gamma) \subseteq KK(A, B)$, the connecting maps are 0, so the cyclic exact sequence gives two short exact sequences

$$0 \to K_0(SB) \cong K_1(B) \to K_0(E) \to K_0(A) \to 0$$

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Which define elements of $\mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), K_1(B))$ and $\mathrm{Ext}^1_{\mathbb{Z}}(K_1(A), K_0(B))$.

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Thus there is a degree one map κ from $ker(\gamma)$ to $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$. If A is in the bootstrap class, κ is a bijection and δ is its inverse.

So if A and B are separable C*-algebras, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(\gamma) & \longrightarrow & KK^*(A,B) & \stackrel{\gamma}{\longrightarrow} & image(\gamma) & \longrightarrow & 0 \\ & & \kappa \downarrow & & \downarrow id & & \downarrow \iota & & \\ & & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) & & KK^*(A,B) & \stackrel{\gamma}{\longrightarrow} & \mathrm{Hom}(K_*(A), K_*(B)) & & \end{array}$$

with the first row exact. The map ι is injective (by definition), but κ is not obviously either injective or surjective.

The UCT holds for a pair (A, B) if and only if all three of the following are true:

The map ι is surjective (i.e. γ is surjective).

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Failure of surjectivity of either ι or κ means $KK^*(A, B)$ is "smaller than the UCT would predict."

Failure of injectivity of κ means $KK^*(A, B)$ is "larger than the UCT would predict."

So we can reduce the UCT to three questions. Let A and B be separable C*-algebras, with A nuclear. (We usually ask these questions for a fixed A, letting B vary.)

Question 1:

Is ι always surjective, i.e. is γ always surjective?

Question 2:

Is κ always injective?

Question 3:

Is κ always surjective?

So we can reduce the UCT to three questions. Let A and B be separable C*-algebras, with A nuclear. (We usually ask these questions for a fixed A, letting B vary.)

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Is ι always surjective, i.e. is γ always surjective?

Question 2:

Is κ always injective?

Question 3:

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These questions are not independent: for example, for a given A, if (1) and (2) have positive answers for all B with $K_*(B)$ divisible, then all three have positive answers for all B (Rosenberg-Schochet).

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$K_*(A)$ and $K_*(A^c)$ are naturally isomorphic since the projections, unitaries, and partial isometries in A^c are just the projections, unitaries, and partial isometries in A respectively. Thus if ι is surjective, there is an $\mathbf{x} \in KK(A, A^c)$ with $\gamma(\mathbf{x})$ an isomorphism.

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Proposition (Rosenberg-Schochet):

Let A and B be separable C*-algebras with $A \in \mathcal{N}$. Let $\mathbf{x} \in KK(A, B)$. If $\gamma(\mathbf{x}) \in \mathrm{Hom}(K_*(A), K_*(B))$ is an isomorphism, then $\mathbf{x}$ is a KK-equivalence.

So if $A \in \mathcal{N}$, then A and A^c (or A^{op}) are KK-equivalent.

If A is a separable nuclear C*-algebra which is not KK-equivalent to A^c, the Proposition shows that at least one of Questions (1) – (3) has a negative answer for some pair (A, B). Which pair(s) and which question(s)?

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The first guess might be that Question (1) has a negative answer (i.e. γ is not surjective) for (A, A^c), although this is not clear. In fact, if there is a separable nuclear A with trivial K-theory which is not KK-equivalent to A^c, then it must be that $KK(A, A)$ is not 0, i.e. κ is not injective for the pair (A, A).

The proof of the Rosenberg-Schochet Proposition also shows:

Proposition:

Let A and B be separable C*-algebras such that the UCT sequence holds for (A, A) and (A, B). Let $\mathbf{x} \in KK(B, A)$. If $\gamma(\mathbf{x}) \in \mathrm{Hom}(K_*(B), K_*(A))$ is an isomorphism, then $\mathbf{x}$ is a KK-equivalence.

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Applying this to $B = A^c$, we obtain that if A is not KK-equivalent to A^c, then at least one of the three questions has a negative answer for either (A, A) or (A, A^c).

Construction of a counterexample to the Main Question may be difficult. If $A = C^*(X, n)$ with $X = L(7, 2)$, then A is not symmetric. But, like all Type I C*-algebras, this A is KK-equivalent to A^{op}; there is a homotopy result about lens spaces which "explains" this ($L(7, 2)$ and $L(7, 3)$ are oriented homotopy equivalent).

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No counterexample can be constructed simply with groups, or more generally with C*-algebras which are the complexifications of real C*-algebras. But perhaps a more sophisticated construction involving cocycles could yield an example.

A counterexample might be constructed as a crossed product by a finite cyclic group, as in Connes' factor examples. It is not known whether the Bootstrap Class is closed under crossed products by finite cyclic groups; in fact:

Theorem.

If $\mathcal{N}$ is closed under crossed products by $\mathbb{Z}_2$, then $\mathcal{N}$ is the class of all separable nuclear C*-algebras.

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Theorem.

If $\mathcal{N}$ is closed under crossed products by $\mathbb{Z}_2$, then $\mathcal{N}$ is the class of all separable nuclear C*-algebras.

It is also known that if $\mathcal{N}$ is closed under crossed products by $\mathbb{T}$, then it is the class of all separable nuclear C*-algebras.

Real KK-Theory

Proposition:

Let A and B be separable (complex) C*-algebras. Then

$$KK_{\mathbb{R}}(A, B) \cong KK(A, B) \oplus KK(A^c, B)$$

In particular, if A is a separable complex C*-algebra, $KK_{\mathbb{R}}(A, A)$ is isomorphic to $KK(A, A) \oplus KK(A^c, A)$. $KK_{\mathbb{R}}(A, A)$ is a $\mathbb{Z}_2$-graded ring with $KK(A, A)$ the degree 0 part. (This is a different grading than the grading of $KK^*_{\mathbb{R}}(A, A)$ by degrees, which is a $\mathbb{Z}_8$-grading in the real case, although it collapses to a $\mathbb{Z}_2$-grading if A is complex).

If the UCT fails because there is a separable nuclear A which is not KK-equivalent to A^c, it is conceivable that there could be a substitute UCT involving real KK-theory, taking into account the nontriviality of the involution on **KN**.

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There is a UCT for real C*-algebras due to J. Boersema. However, this sequence gives no additional information for complex C*-algebras: if A and B are separable complex C*-algebras, Boersema's exact sequence applies only if $A \in \mathcal{N}$, in which case it essentially gives two copies of the Rosenberg-Schochet UCT sequence.

One last observation. If A is a separable C*-algebra, then $KK(A, A)$ is a ring. The set of finite sums of elements of $KK(A, A)$ which factor through A^c, i.e. are of the form $\mathbf{xy}$ for some $\mathbf{x} \in KK(A, A^c)$ and $\mathbf{y} \in KK(A^c, A)$, is an ideal in $KK(A, A)$, which is all of $KK(A, A)$ if A and A^c are KK-equivalent.

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What information does the quotient ring give?