C*-Algebras Generated by Arrays of Projections

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Origin: talks by S. Vaes, R. Vergnioux, T. Banica at Oberwolfach, August 2005

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If $m, n \geq 2$, let Q_{mn} be the universal C*-algebra generated by an array

$$\{q_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

of projections satisfying the *array condition*:
For each fixed i, or for each fixed j, the projections $\{q_{ij}\}$ are mutually orthogonal.

For Q_{mn}, the array condition is equivalent to:
The $m \times n$ matrix

$$B = \begin{bmatrix} q_{11} & \cdot & \cdot & \cdot & q_{1n} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ q_{m1} & \cdot & \cdot & \cdot & q_{mn} \end{bmatrix}$$

is a partial isometry (i.e. B^*B is a projection in $M_n(Q_{mn})$ and hence BB^* is a projection in $M_m(Q_{mn})$.)

Let Q^r_{mn} be the universal unital C*-algebra generated by an array

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of projections satisfying the array condition and the additional condition

$$\sum_{j=1}^{n} q_{ij} = 1$$

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Proposition.

If $\{q_{ij}\}$ is an array of projections on a nontrivial Hilbert space satisfying the relations in Q_{mn}^r, then

(i) $m \leq n$.

(ii) If $m = n$, then $\sum_{i=1}^{n} q_{ij} = 1$ for all j.

Write $A_s(n)$ for Q^r_{nn}. $A_s(n)$ is the universal unital C*-algebra generated by an $n \times n$ array $\{q_{ij}\}$ of projections satisfying the array condition and such that each row and column sum to 1.

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Equivalently: $A_s(n)$ is the universal (necessarily unital) C*-algebra generated by an $n \times n$ array $\{q_{ij}\}$ of projections such that

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is a unitary (or just an isometry) in $M_n(A_s(n))$.
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$A_s(n)$ is finite-dimensional and commutative for $n = 2, 3$, and infinite-dimensional and noncommutative for $n \geq 4$ (discussed later).

Abelianization of $A_s(n)$

Let $A_s(n)_{ab}$ be the abelianization of $A_s(n)$.

Theorem.

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Regard S_n as the set of $n \times n$ permutation matrices. $C(S_n)$ has $n!$ minimal projections $\{p_\sigma : \sigma \in S_n\}$. For each (i,j), set

$$p_{ij} = \sum_{\sigma_{ij}=1} p_\sigma .$$

The map $q_{ij} \mapsto p_{ij}$ is an isomorphism from $A_s(n)_{ab}$ to $C(S_n)$.

Thus $A_s(n)$ can be regarded as a "noncommutative symmetric group." It is, in fact, a compact quantum group, the universal quantum group of quantum symmetries of $\{1, 2, \ldots, n\}$ (S. Wang). The comultiplication is given by

$$\mu : A_s(n) \to A_s(n) \otimes A_s(n)$$

$$\mu(q_{ij}) = \sum_{k=1}^{n} q_{ik} \otimes q_{kj}$$

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Remark: The above formula also gives a comultiplication on Q_{nn}, making Q_{nn} into some sort of "quantum group," which is noncompact since Q_{nn} is nonunital. The range of μ appears to be "too small" to make Q_{nn} a true quantum group. Although Q_{nn} is nonunital, the comultiplication is a homomorphism from Q_{nn} into $Q_{nn} \otimes Q_{nn}$, not just into the multiplier algebra.

Preliminary Observations

$Q_{mn} \cong Q_{nm}$; so may assume $m \leq n$ (automatic for Q^r_{mn})

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Any permutation of the rows or of the columns of the array gives an automorphism of Q_{mn} or of Q^r_{mn} or $A_s(n)$ ($S_m \times S_n$ acts as automorphisms of Q_{mn} etc.) Transpose also gives an automorphism of Q_{nn} or of $A_s(n)$.
In particular, for any i, j, k, l, q_{ij} is conjugate to q_{kl} under an automorphism of Q_{mn} (Q^r_{mn}, $A_s(n)$).

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In particular, for any i, j, k, l, q_{ij} is conjugate to q_{kl} under an automorphism of Q_{mn} (Q^r_{mn}, $A_s(n)$).
But note that q_{ij} is not equivalent to q_{kl} in Q_{mn} (Q^r_{mn}, $A_s(n)$) unless $i = k$ and $j = l$ (discussed later).

Relations Between the Algebras

If $m_1 \leq m$ and $n_1 \leq n$, the C*-subalgebra of Q_{mn} generated by

$$\{q_{ij} : 1 \leq i \leq m_1, 1 \leq j \leq n_1\}$$

is called a *subarray subalgebra* of Q_{mn} of size (m_1, n_1). Similarly for $A_s(n)$.

Proposition.

A subarray subalgebra B of Q_{mn} of size (m_1, n_1) is isomorphic to Q_{m_1,n_1}.

There is an obvious homomorphism ϕ from Q_{m_1,n_1} onto B. There is also a homomorphism ψ from Q_{mn} to Q_{m_1,n_1} sending q_{ij} to q_{ij} if $1 \le i \le m_1$, $1 \le j \le n_1$ and sending q_{ij} to 0 if $i > m_1$ ir $j > n_1$. $\psi \circ \phi$ is the identity on Q_{m_1,n_1}.

So Q_{m_1,n_1} naturally embeds in Q_{mn}, and there is a retraction from Q_{mn} onto Q_{m_1,n_1}.

There is a similar homomorphism ϕ from Q_{mn} onto a subarray subalgebra of $A_s(k)$ for $k \geq \max(m, n)$, which is not injective in general. There is a homomorphism ψ from $A_s(n+m)$ onto $\tilde{Q}_{mn}$ defined by

$$\begin{bmatrix} q_{11} & \cdot & \cdot & \cdot & q_{1n} & p_1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ q_{m1} & \cdot & \cdot & \cdot & q_{mn} & 0 & \cdot & \cdot & \cdot & p_m \\ r_1 & \cdot & \cdot & \cdot & 0 & q_{11} & \cdot & \cdot & \cdot & q_{m1} \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & r_n & q_{1n} & \cdot & \cdot & \cdot & q_{mn} \end{bmatrix}$$

where $p_i = 1 - \sum_{j=1}^{n} q_{ij}$, $r_j = 1 - \sum_{i=1}^{m} q_{ij}$.

Then $\psi \circ \phi$ is the identity on Q_{mn}, so $\phi : Q_{mn} \to A_s(m+n)$ is injective.

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Question:

What is the smallest k such that the map $\phi : Q_{mn} \to A_s(k)$ is injective?

We have $\max(m+2, n+2) \leq k \leq m+n$.

There is no obvious embedding of $A_s(m)$ into $A_s(n)$ for $m < n$. But there is a homomorphism from $A_s(n)$ onto $A_s(m)$ given by

$$\begin{bmatrix} q_{11} & \cdot & \cdot & \cdot & q_{1m} & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ q_{m1} & \cdot & \cdot & \cdot & q_{mm} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

and thus $A_s(m)$ is a quotient of $A_s(n)$ for $m < n$.

Structure of Q_{22}

Q_{22} is generated by 4 projections

$$\{q_{11}, q_{12}, q_{21}, q_{22}\}.$$

The relations are $q_{11} \perp q_{12}, q_{21}$ and $q_{22} \perp q_{12}, q_{21}$. There are no relations between q_{11} and q_{22}, or between q_{12} and q_{21}.

Thus the positive elements $q_{11} + q_{22}$ and $q_{12} + q_{21}$ are orthogonal, and Q_{22} is the direct sum of the hereditary C*-subalgebras they generate. Each hereditary subalgebra is isomorphic to $\mathbb{C} * \mathbb{C}$ (the universal C*-algebra generated by 2 projections).

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So Q_{22} is infinite-dimensional and noncommutative, but 2-subhomogeneous. $Q_{22} \subseteq Q_{mn}$ for all $m, n \geq 2$, so all Q_{mn} are infinite-dimensional and noncommutative. Since $A_s(4)$ contains a C*-subalgebra isomorphic to Q_{22}, $A_s(4)$ is also infinite-dimensional and noncommutative. If $n > 4$, $A_s(4)$ is a quotient of $A_s(n)$, so $A_s(n)$ is also infinite-dimensional and noncommutative.

Structure of $A_s(4)$

Banica: $A_s(4)$ has many 4-dimensional representations, most of which are irreducible. Specifically, there is a homomorphism ρ from $A_s(4)$ to $C(SU(2), \mathbb{M}_4)$ which is "inner faithful."

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It does at least seem plausible that every irreducible representation of $A_s(4)$ is finite-dimensional, so $A_s(4)$ is Type I (CCR).

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Theorem.

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Lemma.

The C*-subalgebra of Q_{23} generated by $\{q_{11}, q_{22}, q_{13}\}$ is isomorphic to the universal C*-algebra B generated by three projections p, q, r with the relation $p \perp r$.

There is an obvious homomorphism ϕ from B to the C*-subalgebra. There is also a homomorphism $\psi : Q_{23} \to B$ sending q_{11} to p, q_{22} to q, q_{13} to r, and the other generators to 0; $\psi \circ \phi$ is the identity on B.

Lemma.

There is a surjective homomorphism from $\tilde{B}$ to $C^*(\mathbb{Z}_3 * \mathbb{Z}_2)$.

The map sends p and r to two of the spectral projections of the $\mathbb{Z}_3$ generator, and q to a spectral projection of the $\mathbb{Z}_2$ generator.

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Since $C^*(\mathbb{F}_2) \subseteq C^*(\mathbb{Z}_3 * \mathbb{Z}_2)$, $\tilde{Q}_{23}$ contains a C*-subalgebra with $C^*(\mathbb{F}_2)$ as a quotient, and thus is not exact. So Q_{23} is not exact. If $m + n \geq 5$, then Q_{23} embeds in Q_{mn}, so Q_{mn} is not exact.

Since $Q_{23} \subseteq A_s(5)$, which is a quotient of $A_s(n)$ for any $n > 5$, we obtain:

Corollary.

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Related result:

Theorem

(Banica). The discrete quantum group associated with $A_s(n)$ for $n \geq 5$ is not amenable.

This is possibly a noncommutative version of the fact that S_n is not solvable for $n \geq 5$, and should mean that $A_s(n)$ is not nuclear for $n \geq 5$.

K-Theory

If the full free product $\mathbb{C}^{*mn}$ of mn copies of $\mathbb{C}$ is regarded as the universal C*-algebra generated by mn projections, and the finite-dimensional commutative C*-algebra $\mathbb{C}^{mn}$ is regarded as the universal C*-algebra generated by mn mutually orthogonal projections, there are natural homomorphisms

$$\mathbb{C}^{*mn} \xrightarrow{\phi} Q_{mn} \xrightarrow{\psi} \mathbb{C}^{mn}.$$

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Theorem.

The maps ϕ and ψ are stable homotopy equivalences. So Q_{mn} is KK-equivalent to the finite-dimensional commutative C*-algebra $\mathbb{C}^{mn}$.

$\psi \circ \phi : \mathbb{C}^{*mn} \to \mathbb{C}^{mn}$ is a *KK*-equivalence (Cuntz). Examination of Cuntz's proof, checking that the array relations are satisfied at a crucial point, shows that it applies verbatim to show that ψ is a *KK*-equivalence, and in fact a stable homotopy equivalence.

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Corollary.

$K_0(Q_{mn}) \cong \mathbb{Z}^{mn}$ as ordered groups, and

$$\{[q_{ij}] : 1 \leq i \leq m, 1 \leq j \leq n\}$$

are free generators of $K_0(Q_{mn})$); the q_{ij} are minimal projections in Q_{mn}. Also, $K_1(Q_{mn}) = 0$.

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So, although Q_{mn} is not exact unless $m = n = 2$, it is K-theoretically tractable.

The K-theory of $A_s(n)$ is (apparently) more difficult to compute. We have only partial information. Consider the case $n = 4$. The 16 elements $\{[q_{ij}] : 1 \leq i, j \leq 4\}$ in $K_0(A_s(4))$ are not independent; there are 3 (independent) relations from the equality of the row sums, and 4 more (one redundant) relations from the equality with the column sums. Thus the group they generate has rank at most 10, generated by the 10 elements

$$\{[q_{ij}] : 1 \leq i, j \leq 3\} \cup \{[q_{14}]\}.$$

A brute-force calculation shows that the images of these elements in $K_0(A_s(4)_{ab}) \cong \mathbb{Z}^{24}$ are linearly independent and generate a direct summand. Thus they generate a subgroup of $K_0(A_s(n))$ isomorphic to $\mathbb{Z}^{10}$ which is a direct summand. We may conjecture that this is all of $K_0(A_s(4))$.

Similarly, the obvious subgroup of $K_0(A_s(n))$ has n^2 generators and $2n-2$ relations, so $K_0(A_s(n))$ should have a direct summand of rank

$$n^2 - 2n + 2 = (n-1)^2 + 1.$$

This subgroup is the image of $K_0(\tilde{Q}_{n-1,n-1})$ under the natural homomorphism corresponding to mapping $Q_{n-1,n-1}$ onto a subarray subalgebra.

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Conjecture.

If $n \geq 4$, then $K_0(A_s(n)) \cong \mathbb{Z}^{n^2-2n+2}$ and $K_1(A_s(n)) = 0$.

A note of caution, though: the formula is false for $n = 3$. It can be argued that this is an exceptional case. The formula is true for $n = 2$, which should also be an exceptional case.

Semiprojectivity

Definition.

A separable C*-algebra A is *semiprojective* if, for any C*-algebra B, increasing sequence $\langle J_n \rangle$ of (closed two-sided) ideals of B, with $J = [\cup J_n]^-$, and *-homomorphism $\phi : A \to B/J$, there is an n and a *-homomorphism $\psi : A \to B/J_n$ such that $\phi = \pi \circ \psi$, where $\pi : B/J_n \to B/J$ is the natural quotient map.

Proposition:

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Proof: If $\{q_{ij}\}$ are projections in B/J satisfying condition for Q_{22}, then $c = q_{11} + q_{22}$ and $d = q_{12} + q_{21}$ are orthogonal positive elements in B/J. Lift these to orthogonal positive elements $a, b \in B$ by Loring or as follows: let $y = c - d$. Then $y = y^*$; let $x = x^*$ be a preimage in B, and let $a = x_+$, $b = x_-$. Replace B by $\overline{aBa}$, etc., and lift q_{11} and q_{22}; then replace B by $\overline{bBb}$ and lift q_{12} and q_{21}.

What about Q_{23} or more general Q_{mn}?

For Q_{23}, can partially lift any 5 of the generators with the right relations, but hard to lift the last one.

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Cause for optimism: same type of difficulties seem to arise in trying to show directly that $O_m \otimes O_n$ is semiprojective $(2 \leq m, n \leq \infty)$. But many, maybe all, of these are semiprojective.

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What about $A_s(n)$? Easy to show: if Q_{nn} is semiprojective, then $A_s(n)$ is semiprojective. Converse is unclear.